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VOLUME I.
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at ARMOUR INSTITUTE of TECHNOLOGY

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INTRODUCTION BY THE EDITOR-IN-CHIEF.

Recent developments in engineering education in America have apparently amazed many thoughtful persons. The gift of a great engineering laboratory to Stevens Institute by so untheoretical a man as Mr. Andrew Carnegie, the recent addition to the endowment and efficiency of Purdue University, the rearing of a new building for shops at the Armour Institute of Technology, and many other such events, within one brief year, are indications which may easily confuse rather than illuminate the average man. But what they do indicate is just this: That the United States is awake to the fact that a future whose portals have been so recently won by our conquest of trade and commerce through technical achievements is to be held and developed only by larger and more thorough technical education.

Religion builded her schools in other ages for the development of the heart, and all the emotions of worship were cultivated; then came the great series of schools which educated the head, and man's intellectual life has been wonderfully exalted and broadened and vivified. This age of ours, without neglecting head or heart, beholds this same world as so much material to be transformed and utilized toward man's weal. It is all ore of some sort, to be smelted, refined, and at length minted, as it were, by man's hand, so that its powers and resources shall be current coin. Thus the technical school of to-day is as admirable and costly a product as any so valuable an evolution. It has been a long time coming, and it is not strange that it should have a very lofty development in America.

We demand it, and we demand something more. Our resources, in the first place, are such as require broad and deep education even for their slightest use, and surely for their fullest alliance with the hopes of mankind. Great fields of coal, mountains of iron, and vast harvests to be hauled from the field to the market have demanded railways, civil engineers to build them, and railway engineers to make the rolling stock efficient. In the same

way our gulches, gleaming with gold and silver, or colored with the deposits of copper found in the neighborhood of other natural wealth, have created the necessity for the mining engineer. In this way engineers of all sorts have been called for by the very riches of nature. The chemist and the electrician are now engineers, because, as we perceive in this first look at the subject, there is a pressure upon them, in the name of civilization, to do something adequate with the raw material in nature. Then the needs of man, increasing with civilization, coming from his many-sided and deepened life, will not be satisfied with a cursory or amateurish handling of nature. Man demands that the things he gets on with, in the form of tools and instrumentalities, shall be equal in their intellectual import with his books and his songs and his commerce.

The man who has done things with nature by rule-of-thumb is thus pushed aside by the man who is not trying experiments with nature and her resources. This trained man is always approaching her in perfect confidence of intelligence. He knows that what he has found in the laboratory and the mathematical calculation will be discovered as a permanent habit of nature when he seeks to make a machine. The scholar in engineering has come to stay, just because he turns out to be more practical than a brilliant ignoramus. It will not do for a man to belittle his acquirements in the direction of theory. The theories as to iron and coal and steam, as they are gathered in a boiler and engine, are expressions of laws and methods of nature which he has to count on all the time, lest he shall have nature against him and his machine shall break down. No practical man in any steel mill or railway plant now thinks of laughing at theory. I am not going too far when I say that a lawyer can more easily be a lawyer without a law school, and he can get on by influence of rare genius without much scholarship; even a doctor can do the same, more easily than one who desires to achieve worthy things in mechanical, electrical and chemical work can get on without a technical training.

The American engineering student is flushed with triumph and is thrilled with the new opportunities before him. Within ten

years he has seen America succeed to premiership on lines of his own culture and effort. But there is more ahead in every direction. He hardly knows whether to go back into the technical school and ask his instructors for more years of training before he goes out at all, or to leap forward and continue the line of advance which his fellows have inaugurated. On the one hand, he is convinced that raw material and the primal and insistent forces of nature have been but crudely handled hitherto; on the other, he is fascinated by the fact that, even with his present possessions in knowledge and skill, the conquest is easy and sure. This is all true, if he looks upon the matter only as a competition evidently led by the technically trained American. But there is something more in it all, as we say, than the fact that as France and Germany won over England, so America has outrun the victors themselves in entering all markets and furnishing the highest sort of product.

That something is this: Just at the time when America sees that she has not yet opened her most valuable resources at home, she is confronted with a sudden and well-nigh overwhelming gift of opportunity in her new realm abroad. That new realm is not, as we know in these days, to be subdued by armies with cannon and sword; it is to be brought safely and loyally in league with our American position by the conquest of ideas and ideals. Among the leaders of such an advance, the engineering student in this our practical age is likely to be foremost. Surely no educational scheme can neglect to furnish him with knowledge and with skill, whether the American remain at home or go abroad to do his work; and surely least of all can the young American afford to forego the knowledge and skill which will give him mastery here or elsewhere.

One of the things most unfortunate either to the development of our American resources, or to the development of engineering as a science, or to the man who has been under a misapprehension with regard to how much or how little he can know and do in the world, is the fact that engineering science and practice has been regarded as a field to be entered only by highly trained and favored students. To be masterful in the highest degree in anything is, of course, to be a trained scholar in that thing; but it is simply false

to say that the door of entrance is closed. The conceit of pedantry is laughable enough; but it is worse in its effects than we suppose. To all three of these interests just mentioned nothing has been more disastrous. The current opinion of a few years ago, that a man without certain technical information, difficult to obtain, could be nothing but a hired hand, has done vast harm. It has placed engineering where such a notion would have placed politics, had not the notion been broken to fragments by undeniable facts. Suppose, for example, that the science of politics had been hedged about with as much of academic propriety as has fairly shut out many strong men from intelligent engineering practice, at least so far as the notions of ten years ago were concerned. Here comes along a young man with a burning pine knot and a courageous soul, who has a few books and a clear head. His name is Abraham Lincoln. Now the subject of politics is fully as profound and complex as the subject of engineering. This man gets at the essential, and, by going right at the heart of his subject, the poor boy of the cabin rises to be the most amply equipped and comprehensive master of politics in his time. It has often been said that such men as Lincoln have so much genius that schools would make very little difference with them. It is to be replied that just such young men as Lincoln must have the opportunity presented to them and just such an opportunity must have a Lincoln presented to it, through books or through schools,— it matters not, — in order that, first, we may have full development of our America; second, that we may have the full development of our politics; and third, that we may have the full development of our Lincoln.

So of engineering. Every Lincoln demonstrates that there is more to be acquired and used by what is called the unfavored, and often the ordinary man, than we have agreed upon in our philosophy of education. If he be an extraordinary mind, as was Lincoln, born and reared at the bottom of the ladder, so much the more important for the profession which is hobbling along without him, and so much more important for the nation who needs his remarkable power, that he and his powers have brought to them all that can possibly be known and acquired just where he is.

We have missed too many men who were just a little less than

strong enough to do without what they should have had. Our educational system has been too much like a pyramid resting upon its apex. To save the cause of republicanism and democracy, it must rest upon its base. Education needs an Edison as badly as the young Edison ever needed education, which at last he attained through great sacrifice.

The case which I have used as an illustration has made wise educators ask: "Who is the educated man?" President Pritchett remarks as follows, in his article on "Industrial and Technical Training":

"Unfortunately, notwithstanding two thousand years of discussion, no criterion has been invented by whose application it may be determined if a man be entitled to be called an educated man. It is rather by the absence of certain qualifications than by their possession that this test can be applied. Perhaps it would be generally admitted that no man may be fairly termed an educated man until he can read and write his mother tongue with ease and facility, nor until he has some acquaintance with, and has developed some taste for, the best literature of his own country. Judged by even so modest a standard, it seems probable that a large proportion of the graduates of our colleges and scientific schools of to-day are not educated men. One finds amongst these graduates a large number to whom the colleges have brought education without training, and a large number to whom has been brought training without education. The two do not always go together. Charles Sumner was a better trained man than Abraham Lincoln. He was not so well educated."

Now is it not possible to have our engineering Lincoln both trained and educated? He has his employment for practice, shall he not have something else? At least, if he cannot go to an institution of learning, we can take the institution of learning to him. Such is the purpose of a series of books like this, and such is the purpose of the correspondence work with which its authors and editors are allied.

Whether our professional schools agree to it or not, democratic forces are slowly, where they are not swiftly, making the transformation. Reference libraries, the correspondence schools

of the country and the processes of university extension, Chautauqua movements and the like, are proving that the man least favored by circumstances, if he have a strong purpose and wise guidance, can in time demonstrate how foolish, if not how wicked, have been many of the academic restraints put upon engineering education. The man at the forge, the pattern-maker, miner, trainman and dynamo tender, with all his brethren at labor, is offered elementary education of a quality surprisingly high, according to the best opinion. This new day for the man who has been so long told that he cannot obtain a share of scholarship on the subject in which he works and in which, therefore, he is most interested, has dawned, not entirely because some one has desired to make money or because some one else has wished to appear in the gown of the teacher. It has come with the consent and by the help of the most intelligent, as well as the most philanthropic, minds of our day.

The day is not far distant when large endowments will be given for schools of correspondence; and they will be related to the institutions of learning who have not already schools of correspondence of their own, so that the man at the top, so-called, who knows the most and highest things, shall be connected beneficently, through a system of popular education, which is at once scholarly and clear, with the man at the bottom. The noblest as well as the greatest question which the most richly endowed and magnificently equipped temple of scholarship can ask to-day is: How may we exalt this institution to the very heights of pure and accurate scholarship and investigation, and at the same time find the aspiring man amongst the masses of the people who is desirous of beginning where he is and furnishing his mind and powers with all the culture of which he is capable? It is entirely impossible for scholarship and discovery at the top to accomplish the highest results for itself, in the long run of years, until it gets hold of the ability which springs from the masses, crude and untrained though it be at first, but surest to achieve at the last. The university of to-day comes very close to the least trained and ambitious man, who will realize that Carlyle is right when he says: "The true university of these days is a collection of books." The miracle of our time is found in how near to a firm purpose and a clear head

knowledge may be brought,— not the knowledge of the street, but that knowledge which has hitherto been sacredly or unsacredly guarded as a possession of professional scholars alone.

This series of books, a Library of Modern Engineering Practice, is the product of scholars who have obeyed these convictions thus briefly noted. These pages are the apparatus of teaching, and they have been proved, in thousands of cases, to be understandable by those who have not had the opportunity of attending engineering schools. They have been written with the idea that many a man who otherwise would remain a day laborer all his life shall begin with their use the career of a scholar. Anybody who has a new idea which is true and a new mastery of forces of nature — however little it may be — has entered the career of a scholar. There are no bounds to be placed upon such an investment of time and courage.

Technical Education,

An Economic Necessity.

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The day of the untrained man is past; the day of the technically trained man is here. At no time have the untrained, the unfit, the poorly prepared, been so ruthlessly weeded out of the professions, the arts, the trades and business callings, as they are to-day. The incessant demand, heard by him who has "his ear on the ground," is for young men in the vigor of manhood, whose eyes, ears and hands, as well as minds, are trained to do the work demanded in modern industrial pursuits. Those who stand midway between the active industrial workers on the one hand and the oncoming groups of young men on the other, like the administrative officers and professors of our technical schools, are able to take a broader and more comprehensive view of the field of supply and demand than most others.

The pedagogical reason for the existence of manual training in a school curriculum is identical with the reason for the existence of drawing, singing, number work, science work or reading — no more and no less. To be perfectly fair and impartial, then, it must be admitted that the term "Manual Training School" is as erroneous as "Mathematical Training School," "Science Training School" or "English Training School." The names Primary, Grammar and High School are quite sufficient, for in each school mathematics, English, science and manual training should enter as co-ordinate subjects. Manual training should no more give name to a secondary school than philosophy should be the describing

feature of a university. Rather should shop work, science and hand manipulation be an integral part of the curriculum of all grades of school work, from the kindergarten up. The simple hand work of the kindergarten, the science work of the primary school, sewing and cookery for girls and carpentry for boys in the grammar grades, are all movements in this direction. The manual training of the future, and the very near future, will be hand work of some kind adapted to the ability of the student, whatever may be his stage of advancement. Manual training will then lose the distinctive feature it now holds as an adjunct to a High School course, and become one means of education applicable to all grades of instruction.

Most, if not all, recent improvements in educational methods in this country have been purely academic or philosophic; they have been evolved by musing in the library, or perhaps from classroom experience; but none of them have resulted from a close acquaintance with the real conditions of the industrial and business warfare into which the student enters when he leaves the school. The so-called "systematical development" given to young people is but a twentieth-century way of training a "jack-of-all-trades." The one thing needful will not come from teachers' conferences, nor county institutes, nor the august meetings of school superintendents, for the attendants at these assemblages have not, and cannot get from their experience, the proper point of view. The great majority of the public-school teachers are women whose environment has been limited by the home, the school and some social functions. The number of men engaged in teaching who have done serious practical work outside of the schoolroom is small. The academic circle which they tread reminds one of the Illinois farmer whose circle of experience was to raise more corn, to feed more hogs, to buy more land — to raise more corn, to feed more hogs, to buy more land — *ad infinitum*. Educators of to-day should break out of the circle in which they are traveling and look at matters from a new point of view. This should be from the vantage ground of the factory, the workshop, the railroad or the counting-room; it should be industrial and commercial rather than academic. The practical needs of everyday workers should be considered, after a scientific study of conditions at present existing. There should be no prejudgment by the social or educational philosopher who knows not the difference between a "shaper"

and a "milling machine," and evolves airy theories of what ought to be, but never is. Industrial life in its relation to national growth and prosperity, in its influence on international competition, and in its demands upon educators, offers interesting fields of investigation. This change of view from the speculative and philosophic to the practical and the industrial will result in many desirable changes for the betterment of our system of education. Hitherto the speculative element has been supreme; the time has come for this influence to be tempered by the introduction of more rational and scientific views.

With advancing civilization, with the luxuries of yesterday becoming conveniences to-day and necessities to-morrow, with our products going to far-away lands in competition with local goods, with competition becoming sharper day by day, with the margin of profit becoming steadily narrower, it is found necessary to seek new lines of economy. In the accurate application of the laws of science, an almost limitless field is open. Mere practice, no matter how long continued, cannot long avail against theory followed by intelligent application. A stupid fireman can burn up the profits of his establishment by ignorant firing; an engine man was recently discovered running an engine for no other purpose than to get exhaust steam to heat his building. He was a "practical man of long experience." A soap manufacturer who had depended upon the "knack" of one workman for many years nearly went into bankruptcy when the man died, for no one else seemed to have the requisite "knack"; but the employment of a trained chemist without the "knack" regained for him his lost business. A large railroad company recently saved more than half a million dollars in six months by equipping a laboratory in connection with its machine shops and putting in charge a technically trained man — one who could unite theory with common sense. Day by day the demands for scientific and technical training become more and more emphatic. The present age has truly been called the age of science, but with this designation the story is only half told. Pure science to be of the greatest value must be applied. The application of science to industry has made this the age of technical education; it has changed many of the conditions of life and has given rise to new problems, the solution of which depends in a high degree upon men who have received a technical education.

Technical training, in its broadest significance, should be made to include professional schools, as of law, medicine, theology, engineering, dentistry, pharmacy, the special research work of universities, as well as the more modest schools for the textile industries, the mechanic arts or domestic science. Although professional training may, in this broad sense, be termed technical, yet for the present purpose it will be best to limit its significance and apply it only to the arts and crafts. A dozen years ago the introduction of manual training was always preceded by a spirited controversy; to-day its value is generally conceded and its introduction is becoming fairly common. Not only is it incorporated into the High School curriculum, but it is fast going where it especially belongs — into the graded schools, as an organic part of educational training. In another dozen years the necessity for advocating manual training will probably have ceased. What new features will be advanced to take its place? What new element in educational progress will be demanding recognition? In all probability it will be secondary technical education. Leaving out of consideration the two extremes — unskilled labor at the foot of the educational ladder and those professions at the top which demand a preliminary college training — there are many vocations in the middle ground for which a technical training is necessary. This broad field has been cultivated very little in this country. England, France, Holland, Belgium, Germany, Austria and Switzerland have all done more than we have, and it may not come amiss to examine the system of technical education in one of these countries.

Switzerland is best known to Americans as a region of mountain scenery, but it is an object-lesson for us in a manner not yet fully recognized. The natural conditions under which the Swiss people have struggled for the past six hundred years have been so burdensome and their industrial success so pronounced, that Switzerland is to-day the best of the European countries in which to study the close relation which exists between the technical education of the masses of the people and national industrial prosperity. There are fewer people in all Switzerland than in Greater New York; the largest city — Zurich — contains only a hundred and fifty thousand persons; the area of the country equals the combined area of but three of our smallest states — Massachusetts, Connecticut and Rhode Island; of this area 29 per

cent. is utterly unproductive even for the frugal Swiss; 21 per cent. is forest; 25 per cent. is meadow and grazing land; 13 per cent. under fruit; 12 per cent. is under crops and gardens; the country has no seaport, no navigable river, and no lake of commercial size; she has little or no natural wealth; she must import her raw materials, work them up into salable form and export them; lines of transportation are expensive and rates high; the amount of fertile soil is meager, and would scarcely make a respectable-sized Western ranch. In the United States we have been allowed to blunder at will; to be improvident and wasteful; we cut down our forest with reckless disregard for the future. Twenty-six years ago Switzerland enacted a law that her forest area should never be reduced; and since that time for every tree that has been cut down another has been planted. Forced by almost cruel natural limitations to rigid economy, both of material and labor, compelled to solve the problem of industrial success under exacting requirements, Switzerland has given us a solution which it behooves us to examine with considerable care. That solution is the careful, systematic education of each citizen — be his station in life high or low — so that his efforts will be efficient enough to meet competition in similar lines of work the world over.

Like the United States, Switzerland has no national school system. The constitution of the federation allows the Federal Government to establish a polytechnicum, a federal university and other institutions of higher instruction or to subsidize such local schools as may be selected. Provision for elementary instruction must be made by the several cantons, but the Federal Government in most cases subsidizes these cantonal schools. There is no Federal university, but six cantonal universities exist at Basel (founded in 1460), Zurich (1835), Berne (1834), Geneva (1873), Fribourg (1889), and Lausanne (1891). The polytechnicum at Zurich (founded in 1854) is supported entirely by the Federal Government. This school covers the higher scientific and technical fields in a most thorough manner. For accuracy of instruction, completeness of equipment, and high professional standard, it is the peer of any technical school in Europe. It has the proud distinction of having more of its graduates holding positions of responsibility, and of attracting more foreign students, than any other Continental technical school. With a most complete cement-testing laboratory; a mechanical laboratory covering

every possible demand that can be made upon it; with a department of electricity equipped regardless of expense; with extensive departments of chemistry, forestry and agriculture; with a new department devoted to the testing of building material alone, it is evident that the Federal Government has provided liberally for the most advanced instruction in technical lines. There seems to be no fear in the minds of the Swiss that a man can be too scientific, or that too deep a study of the scientific basis of industrial questions is possible.

Next below the Zurich Polytechnicum are three technica, or technical schools, offering less theory and more pratice, and relating their work more closely to the trades and industries. One is located at Bienne; the second, at Winterthur, is noted all over Europe for the thoroughness of its instruction and the success of its graduates; the third is at Burgdorf, where Pestalozzi taught his first school and where Frœbel at one time was developing his philosophy of education. This group of technical schools gives scientific character to the entire Swiss system of technical education.

The trade and industrial schools — to be carefully distinguished from the group first described — have been established to teach the homely trades, and are a conspicuous feature in the educational and industrial life of Switzerland. They are found in every canton, and are intended to make the local industries more successful. Some cantons with a population not over three hundred thousand have as many as a hundred of these schools, enrolling more than three thousand students.

The watchmaking schools have had a long existence, the Geneva school beginning as early as 1770 with the desire of a journeyman watchmaker, Louis Fargare, to learn more of his trade than the ordinary school instruction provided. Besides the school of watchmaking at Geneva, which has a world-wide reputation, and has contributed largely to the success of the Swiss watches, other schools exist at Locle, La Chaux de Fonds, Neuchatel, Fleurier, Solothurn, Bienne, Porrentruy and St. Imier. In these schools the course covers from three to five years; the age limit is thirteen or fourteen years; vacations are few and brief. At Locle the school is open every day in the year except Sundays and holidays. At Neuchatel three weeks vacation is allowed in summer. The tuition is invariably higher for foreigners than for natives.

At Fleurier the charge is $1.93 per month for natives and $4.83 per month for foreigners. The pupil on entering one of these schools enters into a contract of apprenticeship with the school authorities. At St. Imier a contract must be signed by the parents or guardian. At Porrentruy the pupils are on probation for the first three months, and may be dismissed if found unfitted for the work. At Solothurn pupils must pass a theoretical examination and afterwards a practical test before entering fully upon their apprenticeship. If they leave before the expiration of their contract, a fine of not more than $1.93 per month may be exacted for the unexpired term. The amount of practical work is greatly in excess of the theoretical. At La Chaux de Fonds theoretical instruction is given for only ten hours per week in the first year, twelve hours per week in each of the second and third years, eight hours per week in the fourth, and only seven in the fifth year. All the rest of the time is given to practical work. The object of these schools is primarily to give instruction in every detail of the art of watchmaking. How well they have succeeded all the world knows; but it is not so well known that these schools have converted regions naturally fit for nothing but growing pine-trees, into important industrial centers.

In 1888 a school of metal working was established at Winterthur, as a part of the Industrial Museum, under Federal control. The object was to educate skilled workmen for the various branches of fine metal work. Attendance is in the form of apprenticeship, the course three years in length, and the age limit fifteen years. At Brienz, the present center of the wood-carving industry, a school of wood carving was organized in 1883, under municipal control. Tuition is gratuitous, but a contract of apprenticeship is required. This school has a fully organized faculty; instruction covers not only the art of wood-carving, but the distinctive characteristics of the different woods and the manufacture of all needed tools. Not only are the usual styles of carving studied, but special attention is given to the plants and animals peculiar to the Alpine region, as every tourist in that country knows. No better illustration of the value of technical education can be found than in the straw-plaiting schools like that at Wattwyl. These had a very humble origin, and were established to teach the people in the poorer regions of the sub-Alpine districts a useful occupation. The result has been to create a new

industry and to convert regions naturally poor into flourishing industrial centers. There are also shoemakers' schools, with a peripatetic staff of instructors who give short courses wherever classes can be formed. They also issue publications to the trade, dealing with subjects of technical interest, as the structure of the human foot, or the reasons of the military authorities for the specifications relating to the contracts for boots and shoes for the army. Somewhat different in plan is the school of silk weaving at Wipkingen, near Zurich, which is owned by the Association of Silk Manufacturers of Zurich. Only a few of the graduates of this school are ordinary weavers; most of them hold superior positions, as proprietors, superintendents or master weavers. The School for Carpenters and Shoemakers founded at Berne in 1888 by municipal authority, but aided by Cantonal and Federal grants, is peculiarly interesting. Instruction is gratuitous; hours of instruction are from 7 A.M. to 6 P.M., with a free dinner at noon. The object of the school as published is distinctly utilitarian; viz., to teach thoroughly a trade, to investigate the latest improvements in the trade, and to meet the increasing competition of foreign labor by putting domestic labor on a high plane of efficiency.

The needs of women are by no means neglected. A private school of ladies' tailoring and needlework, with a three-years' course, at Zurich; a trade school at Berne, teaching plain sewing, ladies' tailoring, embroidery, ironing, cookery, housekeeping, mending and making children's garments; a housekeeping school at Worb, which is self-supporting, and has more applications than vacancies; a housekeeping school at Buchs, with a three-months' course, popular with young ladies about to be married; a trade school at Basel, which has enrolled nearly three thousand pupils since its organization in 1879; a school of art and needlework at Zurich, with a curriculum of such artistic, educational and scientific merit as to make it a model for other schools; schools at Berne and Lenzburg to supply the demand for trained and competent household servants. All these emphasize the necessity for giving women as well as men a technical training.

Industrial art schools exist at Geneva and Zurich. The one at Geneva is the oldest in Switzerland, founded in 1751, and is attended chiefly by persons already employed, but wish to devote their spare time to improving themselves in the artistic side of their work. At St. Gallen, the center of the lace industry, a

special school devotes its attention to embroidery and designing. In Zurich alone there are seven schools for woodwork; in Fribourg schools for bookbinding, basket-braiding and stone cutting; in Bienne, a school for railroading; at Waedenschwyl, a school for fruit growers and gardeners; at Zurich, Berne and Neuenburg, schools for theoretical and practical agriculture; at Geneva, a school for horticulture; at Sursee, Brugg and Lausanne, winter agricultural schools; at Ruti, Fribourg and Lausanne, dairy and cheese making schools; at Geneva, Neuenburg, La Chaux de Fonds, Berne and Winterthur, commercial schools. In addition to the already extensive list must be added a large number of institutions designed for the instruction of the humblest members of the working classes. The first of these — a drawing school for apprentices — is now nearly a hundred years old. Between 1820 and 1830 three others were established at Aarau, Baden and Zug; between 1840 and 1850 nine were organized, and between 1850 and 1865 seventy-eight more. Since that time their number has increased with great rapidity, and now reaches into the thousands, while the attendance reaches tens of thousands. Sessions are held evenings and Sundays. The original idea of teaching drawing has been maintained in many, but in others distinct industrial courses in cabinet making, tailoring, upholstering and other trades have been introduced.

Swiss education is the result of no single philosophy, no one method, no selfish utilitarian object; it has not been dominated by social reformers nor by mere theorists; it is a composite type, the result of many conflicting views and necessities. Throughout the entire system can be seen, however, an attempt to solve the problem of the relation between education and industrial prosperity. This point of view is the one thing needful in developing our own system of technical education, and the one idea which we can best borrow from the Swiss. With them no valuable feature — either cultural or technical — is ignored. Each branch of knowledge is prized according to its value as discipline or as a distinct aid in professional or industrial pursuits. Their object is not only to train men and women for their vocations, but to build up new industries or at least to extend and strengthen those already established. They have even enriched the humblest callings with all possible applications of art, skill or science; they have expended money liberally upon every form of technical

education which would increase the workingman's efficiency; they have transformed regions naturally poor and barren into thriving communities; they have entered freely into international competition, and have made the world their market-place; they have shown that technical education is an economic necessity, worth many times its cost. The commercial and industrial success of Switzerland, achieved in the face of great natural obstacles, shows conclusively that the surest foundation for industrial prosperity lies in a complete and thorough-going system of technical education.

Technical education in this country is a product of the last quarter of a century. Before 1875 only a few schools pretended to give a technical education. The first legislation which gave a decided impetus to this form of education for the American youth was the Morrill Land Grant Act of 1862. This Act granted public land to the several States, apportioning 30,000 acres for each Senator and Representative, to be devoted solely to the advancement of instruction in the agricultural and mechanic arts. The full effect of this Act was not felt until after the beginning of the last quarter of the century, when there was established in almost every State a school to carry out the purpose of the Act. From that time on the growth of technological education was remarkable, whether we consider the actual outlay upon material equipment, the increased number of students, the constantly improved methods of instruction, or the continued elevation of the standard of instruction. In addition to the governmental aid rendered, many private citizens subscribed liberally to the needs of institutions already established, or made large endowments for new ones. This great advance came from a general acceptance of the view that industry needed the aid and assistance which comes from technical training. With this change of industrial demands came the needed supply in the form of young men trained, not in the classic halls of some favored university, but in the workshops and the laboratories of institutes of technology. These men are educated to understand the fundamental principles of science, not merely in their broad, general aspect, but in every matter of practice, detail and application. They supply, to only a limited extent, however, the needs of modern industry, since they furnish merely the highest grade of workers, as does the Zurich Polytechnicum and the technica at Winterthur, Bienne and Burgdorf. They furnish a technical training, not for the

many but only for the few. What should be done for the great army of workers in our industries, who need at once the skill of the mechanic, together with sufficient intelligence and training to understand the directions of the professional engineers above them? Their number is legion. At the present time this country makes little or no adequate provision for their instruction. This is the realm of secondary education. Whenever we, as a nation, secure the proper point of view — unhampered by the ultra conservatism of the simon-pure schoolman — and approach the subject as the Swiss have done, from a practical, common-sense, industrial standpoint, and build our industries on the solid foundation of science, as exemplified in technical education. then our industrial supremacy will be assured.

The most ringing note on this subject in recent years was sounded by the Advisory Committee invited by the Trustees of the Carnegie gift to present a plan for the technical school which Mr. Andrew Carnegie proposed to establish in Pittsburg. After considering all phases of the question this committee recommended the organization of three schools to be called respectively, the Carnegie Technical College, the Carnegie Technical High School, and the Carnegie Day and Evening Classes for Artisans. The second of these schools was to be most radical. The Carnegie Technical High School as outlined would extend the scope of the present high school to include, besides the subjects ordinarily taught, engineering principles, steam engine and central station practice, pattern making, machine tool work, blacksmithing, steam fitting, surveying, machine design, instrument making, founding, photography, glass blowing, and such other subjects as the local industries might demand. The establishment of such a school would be the Americanization of the foreign trade and industrial school; if it were copied and made general, it would solve for us the question of secondary technical education.

In considering so radical a measure some important features should be emphasized in order to avoid misapprehension. The main idea in such a plan would be to take a boy just graduating from the grammar school and give him a good general secondary education, together with a training on one or more technical or trade lines. Such a scheme would be unsuccessful if the school were to be maintained on the usual high school lines of to-day; it would necessitate a highly elastic or elective plan of operation,

because no individual could successfully study all the subjects offered. The routine courses in physics, chemistry, mathematics and drawing would necessarily be much shortened; but every progressive teacher knows that much of these subjects, as at present taught, is of little value in strictly technical pursuits. Each one can be revised and reduced to advantage. Much of the laboratory, drafting-room and testing work now done by the technical colleges could be incorporated into the courses, and we should have technical high schools as real "people's colleges." There can be no reasonable doubt that our technical education in the United States is weakest in this spot. In our high-grade institutes of technology we are training the commissioned officers of our industrial army, but the great mass of non-commissioned officers and privates are left uncared for. As indicated in the report of the Carnegie Committee, technical education may be said to run parallel with the general lines of education, and be divided into three parts, corresponding to the grammar, high school and college grades of the established system. The lowest grade is generally picked up in the shop by boys without special training, who have had merely a grammar school education. It is represented in Switzerland by the Sunday, holiday and evening schools for artisans. It develops the "practical man." The highest grade is represented by the polytechnicum and technica in Switzerland and by technical colleges and institutes of technology in this country. These represent the phase of technical education that demands thorough knowledge of mathematics and the sciences, and is distinctly professional in character. Between these two extremes lies the region of secondary technical education which is represented in Switzerland by the trade school and the specialized technical school. In the United States it is represented by more or less feeble efforts in a few trade schools, in a limited number of private institutions aimed to help the workingmen, and in night schools. Besides these instrumentalities there is another, arisen only within the past few years, which for pure bigness eclipses all other efforts — the schools of correspondence. While our educators have been discussing mere theories and methods and have been building a Chinese wall about their so-called superb school systems, shutting out all who were too large to walk through the narrow gateways which they themselves had made, private enterprise has discovered

that half a million young Americans outside the walls were hungry for a technical education. Since they were forbidden to enter the sacred city, technical education was sent out to them. When the history of education shall have been written, no greater reproach will rest upon the educators of this decade than their failure to hear the call of this multitude for an education suited to their needs. Technical education is one of the greatest needs of an enlightened democracy and should not be left to private enterprise. With state normal schools where young men and women can be trained for teaching, with state universities furnishing instruction in agriculture, dairying and stock raising, besides training engineers and librarians — technical education supplied at state or national expense — it is evident that we are irretrievably committed to the policy of providing technical education at public expense. If the farmer's son can secure at the state university free instruction in scientific agriculture, what good reason is there for depriving the son of a mechanic from securing free instruction in engineering practice, machine tool work, or any other industrial calling, at the city technical high school? The answer of the schoolmaster that the public schools are intended not to teach trades but to develop character, citizenship and general culture, is totally unsatisfactory. We must remember that in order to become a good citizen a man must first secure a livelihood by honest toil; that whatever conduces to greater industrial efficiency in the individual increases the national prosperity and indirectly improves citizenship. Technical education does this. Furthermore, if it is proper to train civil engineers, stock raisers, butter and cheese makers, librarians, and the teachers themselves at public expense, why should machinists, engine men, carpenters, plumbers and house painters be excluded? No cleavage line can be drawn. If technical education is a good thing for one class it is equally good for others. Hitherto the ranks of such workers have been supplied by the influx of foreigners trained abroad, aided to some slight extent by apprentices from selected industries. But we should not, as a nation, depend upon any such uncertain supply. We should follow the example of Switzerland and, recognizing the dependence of national prosperity upon technical education, set about the task of providing an education for all classes of workers suited to their callings. The technical high school, if properly equipped and put in

close relationship with the trades and industries, will satisfy this national need; it will not be a copy of the European trade school, but rather an adaptation of the trade school which will be in harmony with American thought and American educational ideals.

Industrial warfare is not a new idea, but its sociological effect in giving impetus to technical education during recent years is noteworthy. We are familiar with martial warfare. Our newspapers for the past few years have been teeming with reports of battles, of warfare, of disease and death as the concomitants of war; we have been called upon to pay homage to the prowess which has conquered our enemies on land and sea; we are playing the part of hero-worshippers as we have not done since the Civil War. But in the midst of all this clamor there has been going on — largely unknown and unnoticed — the most bitter, the most relentless war in the history of the world; a war not for territory, not for naval nor military glory, but for wealth, for industrial supremacy — a contest of brain with brain, skill with skill, economy with economy, technical training with technical training. It is only another example of the "struggle for existence." The war in South Africa or the Philippines is less fatal to the workingman than the ceaseless competition of similar workingmen in other countries. It is this unremitting rivalry between workers of the same class the world over that makes the cultivation of the workingman's powers by means of technical education an absolute necessity. That nation which neglects to equip its workmen with the armament of industry — technical education — will surely be defeated and become a decaying nation. These workers should receive immediate and thoughtful attention, for they are the bone and sinew of successful national and industrial life. That they should be technically trained through the medium of the technical high school is at the present time the greatest economic need of the American people.

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Volume I.

MATHEMATICS

SECTION I.

ARITHMETIC.

PART I.

READ CAREFULLY.

1. **Arithmetic** is a knowledge of numbers and their use in computation.

2. As a basis of our system we have what is termed a **unit**; *i. e.*, one, a single thing; as, for instance, one rivet, one cylinder, one pound or one ton. If we have several of these units, the collection is called a **Number**; thus if we have eight tons or seventy rivets, we have a number of tons or rivets. One is also commonly called a number.

3. The **Unit of a Number** is one of the same kind of units as those units which compose the number. The unit of a number is always one.

4. We call a number **concrete** when its units have reference to particular things; as, for example, five rivets, seven bolts, thirty feet. When a number does not refer to any particular object, it is called **abstract.** If we say six, fourteen, etc., without designating any particular objects, we are using abstract numbers.

5. Numbers are **like** when they designate the same objects, as three girders and seven girders. When unlike objects, such as three girders and seven pulleys, are designated, the numbers are called **unlike.**

6. **Notation** or Expression of Numbers is accomplished in three ways: by words, by figures and by letters. The art of reading figures is called **Numeration.**

7. The common notation is the Arabic, and it has ten different characters or figures, as follows:

0	1	2	3	4	5	6	7	8	9
zero	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*

8. The first character is also called cipher or naught. When unconnected or simply before another expression, it possesses no value. After another figure, or between a decimal point and a figure, it alters the value, as will be shown later on. It denotes the absence of the other characters.

9. By means of these ten characters we can express any number. We accomplish this by grouping them according to the two following principles:

(*a*) Ten units of any order in a number make one unit of the next higher order.

(*b*) The use of the cipher is to fill any place that would otherwise be left vacant.

10. Thus we see that figures may have a **local** value, *i. e.*, dependent on position. If a figure stands alone, it expresses its **simple** value only. If, however, it is changed in position with regard to other figures standing with it, it changes also in value.

This local value is shown in the following table:

Hundred-trillions.	Ten-trillions.	Trillions.	Hundred-billions.	Ten-billions.	Billions.	Hundred-millions.	Ten-millions.	Millions.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Units.	Orders of Units.
6	7	8,	4	9	3,	2	2	0,	6	4	4,	8	1	9,	Figures.
5th group.			*4th group.*			*3d group.*			*2d group.*			*1st group.*			Groups.
Trillions.			*Billions.*			*Millions.*			*Thousands.*			*Hundreds.*			

11. In this table 9 expresses simple value. The other figures have local value and express larger numbers of units. To illustrate: If we place a 1 before the 9, thus 19, the additional value

represented is ten units instead of 1 and as the 9 in the units place has the value of 9 units, the total value of the expression is nineteen units. Let us place an 8 before the 19 thus 819. We now add eight hundred to the number because the 8 is in the hundreds position, and the number is eight hundred and nineteen. We now place a 4 in front of this figure, and, as a result, add four thousand making the number four thousand eight hundred and nineteen. The addition of another 4 increases the number by forty thousand, because it is in the ten-thousands place, and the number denoted becomes forty-four thousand eight hundred and nineteen. The whole number reads six hundred seventy-eight trillion, four hundred ninety-three billion, two hundred twenty million, six hundred forty-four thousand, eight hundred and nineteen.

12. The following table gives the smaller numbers and their names in order to further illustrate the system.

0 zero	10 ten	20 twenty	30 thirty
1 one	11 eleven	21 twenty-one	31 thirty-one
2 two	12 twelve	22 twenty-two	32 thirty-two
3 three	13 thirteen	23 twenty-three	33 thirty-three
4 four	14 fourteen	24 twenty-four	34 thirty-four
5 five	15 fifteen	25 twenty-five	35 thirty-five
6 six	16 sixteen	26 twenty-six	36 thirty-six
7 seven	17 seventeen	27 twenty-seven	37 thirty-seven
8 eight	18 eighteen	28 twenty-eight	38 thirty-eight
9 nine	19 nineteen	29 twenty-nine	39 thirty-nine

40 forty
41 forty-one
etc.
50 fifty
51 fifty-one
etc.
60 sixty
70 seventy
80 eighty
90 ninety
100 one hundred
101 one hundred one
102 one hundred two
etc.

120 one hundred twenty
121 one hundred twenty-one
etc.
130 one hundred thirty
140 one hundred forty
etc.
200 two hundred
201 two hundred one
etc.
300 three hundred
400 four hundred
etc.
1,000 one thousand

1,001 one thousand one
1,002 one thousand two
etc.
2,000 two thousand
etc.
10,000 ten thousand
100,000 one hundred thousand
1,000,000 one million.

13. Another notation, the Roman, is here given, and, though of no particular use in computation, occurs very frequently.

I= 1	XV=15	LXXX= 80
II= 2	XVI=16	XC= 90
III= 3	XVII=17	C= 100
IV= 4	XVIII=18	CXIX= 119
V= 5	XIX=19	CC= 200
VI= 6	XX=20	CCX= 210
VII= 7	XXI=21	D= 500
VIII= 8	XXV=25	DCV= 605
IX= 9	XXX=30	M=1,000
X=10	XXXIV=34	MDL=1,550
XI=11	XL=40	MDCLXVI=1,666
XII=12	L=50	MXCIX=1,099
XIII=13	LX=60	
XIV=14	LXX=70	

In the above when a symbol of lesser value precedes one of greater, that of the former is taken from the latter in order to obtain the value of the two combined. When the symbol of the lesser value follows, that is, is on the right of the greater, the values of the two are to be added together to obtain the value of the combination. If a line is drawn above a symbol *that number of thousands* is denoted.

For example $\overline{X}$ means ten thousand.

14. All arithmetical operations are based on four fundamental processes. They are called addition, subtraction, multiplication and division.

ADDITION.

15. The process of finding a number which is equal to two or more given numbers is called **Addition.** The result thus obtained is called the **Sum** or **Amount.**

16. The sign of Addition is + and reads *plus* (meaning more) ; the sign which shows equality is = and reads *equals.*

To illustrate—a man has four fingers on each hand. The sum of his fingers is eight. We express this in this manner 4 + 4 = 8. That is 4 added to or plus 4 equals 8.

17. Only like numbers can be added. For instance, if you have four dollars and are given three more, you have seven dollars ($4 + $3 = $7), but if you have four dollars and a theatre ticket you cannot say that you have more dollars.

18. The following table gives the sum of any two numbers up to 12.

To use this table take one number in the upper row above the double line and the other in the first column to the left of the double line. Opposite the latter number and directly below the former is the sum of the two. For example, take 7 and 11. Look in the same column with 7 and the same row with 11 and we find 18. Look next in the same row with 7 and the same column as 11 and we also find 18. We express the result thus 7 + 11 = 18.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12	13
2	2	3	4	5	6	7	8	9	10	11	12	13	14
3	3	4	5	6	7	8	9	10	11	12	13	14	15
4	4	5	6	7	8	9	10	11	12	13	14	15	16
5	5	6	7	8	9	10	11	12	13	14	15	16	17
6	6	7	8	9	10	11	12	13	14	15	16	17	18
7	7	8	9	10	11	12	13	14	15	16	17	18	19
8	8	9	10	11	12	13	14	15	16	17	18	19	20
9	9	10	11	12	13	14	15	16	17	18	19	20	21
10	10	11	12	13	14	15	16	17	18	19	20	21	22
11	11	12	13	14	15	16	17	18	19	20	21	22	23
12	12	13	14	15	16	17	18	19	20	21	22	23	24

19. In addition of large numbers the numbers are written so that units are under units, tens under tens, etc. The columns are then added separately. For example, add 476, 841, 57, and 1,867. 6, 1, 7, and 7 are in units column, and must be one above the other.

476
841
57
1,867

21	sum of	first	column	(units)
22	"	" second	"	(tens)
20	"	" third	"	(hundreds)
1	"	" fourth	"	(thousands)
3241	"	" all columns or the amount.		

NOTE. The last figure of the sum of each column must come under that column.

20. The process should be completed in one step, thus:

476
841
57
1,867
3,241

This is done by adding the 2 of the 21 first obtained directly to the 22 of the second, thus getting 24. The 4 is set down in the same way, and the 2 added to the next column and so on.

21. The rules for adding are:

(a). Write the numbers so that figures of the same order stand in the same column, *i. e.*, units under units, tens under tens, etc.

(b). Beginning with the right hand column, add each column separately, and write the sum, if expressed by one figure, under the column added. If there is more than one figure write only the last or units figure and add the remaining figure or figures to the next column.

(c). Write the whole sum of the last addition.

22. To illustrate these rules let us add 341 and 422 and 113.

$$\begin{array}{r} 341 \\ 422 \\ 113 \\ \hline 876 \end{array}$$

The sum of the first column is $3 + 2 + 1 = 6$, of the second $1 + 2 + 4 = 7$, and of the third $1 + 4 + 3 = 8$. The answer is, therefore, eight hundred seventy-six (876).

To illustrate and include Rule *b*, let us take the numbers 893, 768, and 941.

$$\begin{array}{r} 893 \\ 768 \\ 941 \\ \hline 2{,}602 \end{array}$$

Here we have $3 + 8 + 1 = 12$ for the units column, so that 2 is put down in the units place, and the 1 added into the tens column where it belongs, so that we obtain $1 + 4 + 6 + 9 = 20$, and here 0 is written down and the 2 carried to the hundreds place, which being added gives $2 + 9 + 7 + 8 = 26$, which is the last sum and is therefore put down at once.

EXAMPLES FOR PRACTICE.

Find the sum of

(1) $56 + 49 + 17 + 21 + 36$. Ans. 179

(2) $42 + 46 + 34 + 58 + 91$. Ans. 271

(3) $467 + 536 + 84 + 705$. Ans. 1,792

(4) $2{,}008 + 1{,}400 + 706 + 300 + 77$. Ans. 4,491

(5) $8{,}950 + 15{,}765 + 7{,}732$. Ans. 32,447

(6) $26{,}661 + 8{,}735 + 6{,}877 + 33{,}413$ Ans. 75,686

(7) $8{,}972 + 980 + 5{,}607 + 89$. Ans. 15,648

(8) $346 + 4{,}682 + 64 + 798 + 21$. Ans. 5,911

(9) If 4 loads of coal weigh respectively 1,922 pounds, 1,609 pounds, 2,100 pounds, and 1,873 pounds, what is the entire weight. Ans. 7,504 pounds.

(10) Coal is fed to a furnace as follows: Monday 276 pounds, Tuesday 307 pounds, Wednesday 438 pounds, Thursday 425 pounds, Friday 299 pounds, Saturday 201 pounds. Find the total for the week. Ans. 1,946 pounds.

(11) A marine engine, during three hours' run, makes 9,187 revolutions the first hour, 9,062 the second, and 9,233 the third. How many does it make in the three hours?

Ans. 27,482 revolutions.

SUBTRACTION.

23. **Subtraction** is the process of finding how great is the difference between two numbers.

24. The number from which another is to be taken is called the **Minuend.**

25. The number to be taken away from the minuend is called the **Subtrahend.**

26. The number left after taking the subtrahend from the minuend is called the **Difference** or **Remainder.**

27. The sign of subtraction is — and reads *minus.* It means *less.* If we wish to subtract 9 from 11, we write 11—9. We read this, eleven minus nine. Eleven is the minuend and nine is the subtrahend.

28. Illustrative example. Subtract 2,437 from 7,549.

Minuend	7,549
Subtrahend	2,437
Remainder	5,112 Ans.

The process is begun with the right hand column, thus: the 7 is subtracted from the 9 leaving the 2, then 3 is taken from 4 leaving 1, then 4 from 5 giving another 1, and then the 2 from the 7 leaving 5.

29. Sometimes, however, the figures of the subtrahend are larger than those of the minuend. We then borrow one of the preceding units to increase the following one by ten.

For example, subtract 7,856 from 9,754.

9,754
7,856

1,898

Here we see that the 6 is greater than the 4, so we take one from the 5, leaving it 4. This adds ten to the 4, because the 5 is in the place preceding it. Thus the 4 becomes 14, and we may subtract the 6, which gives us the 8, a part of our answer. As we have taken one from the 5 it is now only 4, and we must take one from the 7 to increase it. Thus it becomes 14, and we may subtract the 5 beneath it. This gives us the 9. The 7 in the minuend is now a 6, and in order to increase it we take one

from the 9 (leaving 8), and thus increasing the 6, (left after taking 1 from the 7,) to 16. We now subtract our 8, and 8 is the result. Our last step is to take the 7 from the 8, which was left after taking 1 from the 9. This gives us 1 and completes our answer.

30. Carry out the same reasoning in the examples which follow:

6,784
3,776
———
3,008

NOTE. Only the 4 has to be increased. Ten is always the amount we increase an order of units.

1,000
621
———
379

NOTE. Before taking one from the preceding unit we are obliged to increase the preceding unit itself. This we do by taking one from the unit preceding it, increasing that unit also if necessary.

This process of taking units of a higher order to increase those of lower orders, is further illustrated by the following arrangement of figures.

a	A	b	B	c	C	
1—1	10—1		10—1			figures showing subtractions.
0	9		9		10	numbers of units left in each position.
1	0		0		0	minuend.
	6		2		1	subtrahend.
	3		7		9	remainder.

As the last three figures of the minuend are zeros we are obliged to take a unit from the thousands place. We therefore take the 1 marked a, in order to make the 10 marked A. This leaves a 0 in the thousands place as shown by the 0 below. From the 10 marked A we take the 1 marked b, leaving 9 in the hundreds place. This 1 marked b makes the 10 marked B in the next lower order of units. As before, we leave 9 in this place by subtracting the 1 marked c to make the 10 marked C. We are now able to subtract the units of each order from those of like order above them, and obtain the same result as before, namely, 379.

31. RULES. *(a)* Write the subtrahend under the minuend so that units of the same order may be expressed in the same column.

(b) Begin with the units of the lowest order to subtract, and proceed to the highest, writing each remainder under the line in its proper place.

(c) If any term of the minuend is less than the corresponding term of the subtrahend, add ten to it and then subtract; but consider that the next preceding term of the minuend has been diminished by one.

Proof. Add the remainder to the subtrahend: the sum should equal the minuend.

EXAMPLES FOR PRACTICE.

(1) From 7,282 subtract 4,815. Ans. 2,467

(2) From 64,037 subtract 5,908. Ans. 58,129

(3) From 6,231 subtract 3,084. Ans. 3,147

(4) From 1,740,932 subtract 807,605. Ans. 933,327

(5) From 71,287 subtract 40,089. Ans. 31,198

(6) From 1,000,000 subtract 999,999. Ans. 1

(7) From 104,083 subtract 94,736. Ans. 9,347

(8) From 1,001,001 subtract 909,199. Ans. 91,802

(9) From a tank containing 935 gallons of water, 648 gallons were drawn off. Then 247 gallons ran in. How many gallons were then in the tank? Ans. 534 gallons.

(10) A man purchased 8,983 bricks, but used only 5,362. How many had he left? Ans. 3,621 bricks.

(11) A coal shed contains 8,579 tons. 3,243 tons are taken from it. It then receives 4,112 tons more. After that 1,602 tons are taken out of it. How many tons remain?

Ans. 7,846 tons.

MULTIPLICATION.

32. If we wish to add the same number several times, we are able to use a shorter process, called **Multiplication.** For instance, if we wish to add seven eights together, we multiply 8 by 7.

33. The number we wish to add to itself several times is called the **Multiplicand.**

34. The number which shows the number of times we wish to add it to itself is called the **Multiplier.**

In multiplying 8 by 7, the 8 is the multiplicand and the 7 is the multiplier.

35. The sign of multiplication is $\times$, and it is read *times* or *multiplied by.*

36. The result of multiplication is called the **Product.** Thus, multiplicand $\times$ multiplier $=$ product. Take, for instance, $8 \times 7 = 56$. Here 8 is the multiplicand and 7 the multiplier, and 56 is the product. The product is not affected if the multiplicand and the multiplier change places; *i. e.*, $8 \times 7 = 56$, and also $7 \times 8 = 56$.

37. The following table gives the product of any two numbers which are not greater than twelve. To use the table, take one of the two numbers to be used in the multiplication in the upper row. Take the other in the first column. The product of the two will be found in the same column as (*i. e.*, under) the first, and in the same row with the second. Thus 9 is in the eleventh column and 8 is in the tenth row. Under the 9 and opposite the 8 is found the product, 72. That is, $9 \times 8 = 72$, or $8 \times 9 = 72$. If 0 is one of the two numbers the result is 0, as is readily seen, hence, if any number be multiplied by 0 the product is 0.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

38. If we wish to multiply one number by another we proceed as follows:

376
4

1504

For convenience in multiplying, we write the multiplier under the multiplicand. We then multiply, beginning with the units of the first order, as follows:

$4 \times 6 = 24$. We write down the 4 in the units place, and reserve the 2 to be added to the next product obtained. This next product is $4 \times 7 = 28$. To it we add the 2 thus $28 + 2 = 30$. The 0 is set down, and the 3 reserved to be added to the next product, which is $4 \times 3 = 12$. Adding the 3 we have $12 + 3 = 15$ which, being the last product, is written down. We now have the total product or answer which is 1504.

This is further illustrated by the following:

376
4

24
28
12

1504

By this we see that we multiply the 6 in the units place, putting the result below. We then multiply by the 7 and the product, being in the tens position, takes one place farther to the left, *i. e.*, directly under the 7 itself. In like manner the 12 (the product of 4 and 3) is placed with its units place under the 3. We now add and obtain the answer 1504.

We also see that this process is equivalent to adding 376 four times.

376
376
376
376

1504

39. If the multiplier consists of more than one figure we proceed in the following manner:

cb a
373
234

1492 1st partial product or 4 × 373.
1119 2nd partial product or 3 × 373.
746 3rd partial product or 2 × 373.

87282 Total product or answer.

Let a, b, and c represent respectively the units, tens, and hundreds positions.

We first multiply by 4 as we did in the preceding example, and write the result in the units column, that is under a. This is the first partial product. We then obtain the second partial product by multiplying in the same way by 3, placing the result with its units figure under b, that is in the tens position. In like manner we multiply by 2 and write the third partial product in the hundreds column or under c. Adding these partial products we obtain the total product or answer which is 87,282.

40. If we have one 0 or more in an example, the procedure is practically the same. The following examples illustrate such cases.

624		:	624	
705		:	705	
		:		
3120	product of 5 × 624	:	3120	= 624 × 5
000	product of 0 × 624	:	4368	= 624 × 7 in hundreds
4368	product of 7 × 624	:		place (or 624 × 700)
		:	439920	
439920				

It is customary to contract the work as in the second case. We perceive also by the second case, that (with the last partial product) the real operation is simply multiplying by 700 because we write the result in the hundreds position.

Let us now multiply 624 by 700.

624
700

436800

We here perceive that we get the result in multiplying as we did above in the last partial product. Hence, zeros may be disregarded in multiplying, when they stand at the right hand of other figures, but there must be added to the product the *same number of zeros as were disregarded.*

This applies to *both* multiplicand and multiplier, as is shown by the following examples:

4530	286	2860	59,658
300	100	10	3700
1359000	28600	28600	417606
			178974
			220734600

41. Any number multiplied by zero alone equals 0.

748	1645	21	11	2
0	0	0	0	0
000	0000	00	00	0

42. RULES. *(a)* Write multiplier under multiplicand, units under units, tens under tens, etc.

(b) Each figure of the multiplicand is multiplied by each significant figure of the multiplier, and the right-hand figure of each product is placed under the figure of the multiplier used to obtain it.

(c) The sum of the partial products will be the entire product.

NOTE. When there is a zero in the multiplier, multiply by the significant figures only, taking care to place the right-hand figure of each partial product under the figure used in obtaining it.

Proof. Multiply multiplier by multiplicand. The product should be the same.

43. We must understand that the multiplier is always regarded as an abstract number. The multiplicand and product are like numbers, and may be either concrete or abstract; *i.e.*, if we multiply 5 rivets by 6 we have 30 rivets. We cannot, however, multiply 5 rivets by 6 rivets.

EXAMPLES FOR PRACTICE.

Multiply

(1)	267 by 24.	Ans.	6,408
(2)	918 by 61.	Ans.	55,998
(3)	465 by 27.	Ans.	12,555
(4)	817 by 63.	Ans.	51,471
(5)	837 by 84.	Ans.	70,308
(6)	3,845 by 236.	Ans.	907,420
(7)	5,327 by 420.	Ans.	2,237,340
(8)	4,769 by 344.	Ans.	1,640,536
(9)	2,918 by 364.	Ans.	1,062,152
(10)	7,319 by 394.	Ans.	2,883,686
(11)	5,698 by 792.	Ans.	4,512,816
(12)	8,169 by 734.	Ans.	5,996,046
(13)	3,186 by 839.	Ans.	2,673,054
(14)	3,864 by 609.	Ans.	2,353,176
(15)	42,308 by 692.	Ans.	29,277,136
(16)	Multiply 872 by itself.	Ans.	760,384

(17) There are 746 watts in a horse-power. How many watts are there in 20 horse-powers? Ans. 14,920 watts.

(18) A piston of twelve inches diameter has an area approximately of 113 square inches. If the steam pressure is 47 pounds per square inch, what is the total pressure upon it?

Ans. 5,311 lbs.

(19) The end of a boiler about ten feet in diameter has an area of 11,310 square inches. If the pressure per square inch is 40 pounds, what is the total pressure? Ans. 452,400 lbs.

(20) Two steamers are going in the same direction. One makes 15 miles per hour, the other 11 miles per hour. How far apart are they at the end of 7 hours? At the end of 17 hours?

Ans. 28 miles after 7 hours; 68 miles after 17 hours.

DIVISION.

44. **Division** is the process of finding how many times one number contains another. For example, a foot rule is divided into or contains twelve inches.

45. The number by which we divide is called the **Divisor.** The number divided is called the **Dividend.** The number which shows how many times the divisor is contained in the dividend is called the **Quotient.** If a part of the dividend remains after the divisor is contained an exact number of times, it is called the **Remainder.**

To illustrate, 7 is contained in 74 ten times and 4 is left over. Here 7 is the divisor, 74 the dividend, 10 the quotient, and 4 the remainder.

46. The sign of division is $\div$ and is read *divided by.* If we write $63 \div 7 = 9$, we mean that 63 divided by 7 equals 9. Here 63 is the dividend, 7 the divisor, and 9 the quotient. Another method of writing the same is $\frac{63}{7} = 9$. (See Fractions, Part II.)

NOTE. Division is the reverse of multiplication.

47. Compare the following columns:

$6 \times 7 = ?$	$6 \times ? = 42$	$? \times 7 = 42$	$42 \div 7 = ?$
$5 \times 8 = ?$	$5 \times ? = 40$	$? \times 8 = 40$	$40 \div 8 = ?$
$7 \times 5 = ?$	$7 \times ? = 35$	$? \times 5 = 35$	$35 \div 5 = ?$
$8 \times 3 = ?$	$8 \times ? = 24$	$? \times 3 = 24$	$24 \div 3 = ?$

48. When the divisor consists of a single figure the process of division is carried on in the following manner.

Divisor.	Dividend.	Quotient.
6)	864	(144
	6	
	26	
	24	
	24	
	24	
	0 Remainder.	

In this example we first divide the hundreds by 6 and place the quotient thus obtained in the hundreds place in the quotient.

This shows how many hundred times 6 is contained in 864. After this operation has been performed we place the 6 under the 8 and subtract. We now see that two hundred units remain to be further divided. We add to them the figure in the tens position of the dividend, which, as we see, is 6. These 260 units are now divided by 6. As the next figure of the quotient is to be written in the tens position, we divide the 26 as it stands, thus obtaining the first 4 of the quotient. We now write the 6 multiplied by 4 (24) under the 26 and, subtracting, find that two tens or twenty remain. To this 20 we add the 4 of the units place of the dividend, and complete our quotient by dividing by 6 as before. We find that the 6 is contained four times with no remainder, and we write down the 4 which is the last figure of our quotient. This is called long division.

49. There is a shorter method of doing examples of a simple nature (such as the above). This is called short division.

$$\begin{array}{r} \overset{\text{2 2}}{} \\ 6)864 \\ \hline 144 \end{array}$$

Here we perform the work, for the most part, mentally. We see that 6 will be contained once in 8 and that 2 will be left over. This 2 goes to the next lower order of units, thus producing, with the 6 already there, a value of 26, in which 6 is contained 4 times, and 2 is left over. This second 2 goes to the units place, where it produces a value of 24. Dividing the 24 by 6 we obtain the last 4 of the quotient.

50. If there are two or more figures in the divisor, we proceed as follows:

$$\begin{array}{r} 204)1041829(5107 \\ 1020 \\ \hline 218 \\ 204 \\ \hline 1429 \\ 1428 \\ \hline 1 \quad \text{Remainder.} \end{array}$$

Here 204 is not contained in the first three figures, so we

take the first four. We find that it is then contained 5 times, so we write 5 as the first figure in the quotient. We subtract 5 × 204, or 1020, and find 21 left over. We then bring down the 8, and as 204 is contained once in 218, we place a 1 after the 5 in the quotient. We subtract 204 from 218 and find that 14 remains. We bring down the 2, but find that 204 will not be contained in 142, so that we put a 0 in the quotient and bring down the 9. We then find that 204 is contained (in 1429) 7 times and 1 over, so that the final figure of our quotient is 7.

NOTE. The remainder of 1 may be written as a fraction with the quotient $5107\frac{1}{204}$.

51. RULES. (*a*) Write the divisor at the left of the dividend, separating them by a line.

(*b*) Find the number of times the divisor is contained in the least number of left-hand figures of the dividend that will contain it, and write the result as the first figure of the quotient.

(*c*) Multiply the divisor by this figure of the quotient just found, and subtract the product from the part of the dividend (first partial dividend) just used to contain the divisor. To the remainder found by subtracting, annex the figure of the next lower order of the dividend and then use *it* as a new partial dividend, dividing it by the divisor as before. Continue this process until all the figures in the dividend have been used.

(*d*) If any partial dividend does not contain the quotient write an 0 in the quotient, annex the next figure from the dividend and proceed as before. (If the divisor is not then contained continue to write zeros in the quotient and to annex figures until the partial dividend becomes large enough.)

(*e*) Write the remainder as mentioned in 50—Note.

52. Proof. Multiply the divisor by the quotient and add the remainder. The result will be the dividend.

The following is an example of the proof.

26)5441(209	:	209	
52	:	26	
241	:	1254	
234	:	418	
7 remainder	:	5434	
	:	7	(added)
		5441	

EXAMPLES FOR PRACTICE.

Divide

(1) 65,814 by 6. Ans. 10,969

(2) 614,648 by 8. Ans. 76,831

(3) 3,870 by 10. Ans. 387

(4) 32,000 by 1,000. Ans. 32

(5) 50,854 by 94. Ans. 541

(6) 56,394 by 78. Ans. 723

(7) 559,136 by 202. Ans. 2,768

(8) 2,418,000 by 60. Ans. 40,300

(9) 75,862,500 by 10,115. Ans. 7,500

(10) If coal costs 5 dollars per ton, how many tons may be bought for 275 dollars? Ans. 55 tons.

(11) A steamer runs 276 miles in 23 hours. What is her average speed per hour? Ans. 12 miles per hour.

(12) How many 30 foot rails are there in a double track (four rails) two miles long? There are 5,280 feet in one mile. Ans. 1,408 rails.

(13) If 9 days' work will pay for 6 tons of coal at 6 dollars per ton, what is the price of a day's labor? Ans. 4 dollars.

(14) *Three lots of land were sold for 1,500 dollars, 2,974 dollars, and 1,895 dollars respectively. What was the average price? Ans. 2123 dollars.

NOTE: To find the average of several quantities add them together and divide the sum by the total number of quantities. Thus, if it is required to find the average of six numbers, add them together and divide by six.

FACTORING AND CANCELLATION.

A number is said to be divisible by another, when there is no remainder after dividing.

All numbers are either *odd* or *even*.

An **odd** number is one that is not divisible by 2, as 3, 5, 7, 9, 13, 17, etc.

An **even** number is one that is divisible by 2, as 4, 6, 8, 10, etc.

If an *even* number is divisible by an *odd* number the quotient will be an *even* number.

An *odd* number is not divisible by any *even* number.

A **prime number** is one that is divisible only by itself and one, as 1, 2, 3, 5, etc.

Two (2) is the only **even prime** number since it can be divided by itself and one, and all even numbers are divisible by 2.

A **composite number** is a number that is divisible by numbers other than itself and one, as 4, 6, 8, 10, etc.

A **factor** of a number may be called a divisor and may be used as multiplicand or multiplier to make another number.

All composite numbers are made up of smaller numbers which, when multiplied together, equal that number.

To factor a number is to separate it into its factors, for example: "What are the prime factors of 16?"

$$\begin{array}{r} 2)16 \\ \hline 2)\ 8 \\ \hline 2)\ 4 \\ \hline 2)\ 2 \\ \hline 1 \end{array}$$

We must divide 16 by 2, which is the least prime number greater than 1, and we obtain as our quotient 8. Since 8 is a composite number we divide this also by 2 and obtain as our next quotient 4. Next divide 4 by 2 and we get as our quotient 2. Treat this in the same manner and our quotient is 1, which is a prime number and cannot be further divided. Now the several divisors and the last quotient, all of which are prime, are the prime factors of 16.

$$2 \times 2 \times 2 \times 2 = 16$$

The above method is plainly seen by the following rule.

Divide the given number by the least *prime number* (greater than 1) that will divide it, and each quotient if a *composite number* in the *same manner*. Continue dividing until we obtain a prime number as our quotient. These several divisors and our last quotient will be the prime factors desired.

In order to easily determine the factors of numbers we may use the following table.

TABLE OF PRIME NUMBERS FROM 1 TO 997.

1	41	101	167	239	313	397	467	569	643	733	823	911
2	43	103	173	241	317	401	479	571	647	739	827	919
3	47	107	179	251	331	409	487	577	653	743	829	929
5	53	109	181	257	337	419	491	587	659	751	839	937
7	59	113	191	263	347	421	499	593	661	757	853	941
11	61	127	193	269	349	431	503	599	673	761	857	947
13	67	131	197	271	353	433	509	601	677	769	859	953
17	71	137	199	277	359	439	521	607	683	773	863	967
19	73	139	211	281	367	443	523	613	691	787	877	971
23	79	149	223	283	373	449	541	617	701	797	881	977
29	83	151	227	293	379	457	547	619	709	809	883	983
31	89	157	229	307	383	461	557	631	719	811	887	991
37	97	163	233	311	389	463	563	641	727	821	907	997

EXAMPLES FOR PRACTICE.

Separate the following numbers into their prime factors.

(1) 28.
(2) 87.
(3) 336.
(4) 713.
(5) 914.

CANCELLATION.

In order to save the unnecessary work of multiplying all the numbers above the line, and dividing that product by the product of all the numbers below the line, we have a short way of obtaining the result, by a process called *Cancellation.*

Take for example the following:

$$\frac{6 \times 8 \times 12 \times 18 \times 24}{2 \times 3 \times 2 \times 4 \times 6} = ?$$

In order to cancel, take any two numbers, one above and one below the line, and find some number that will be contained an even number of times in both of them. For instance, take the

number 6 above the line, and the number 2 which is below the line, and we find that 2 will be contained an even number of times in both 6 and 2. We then perform division, crossing out both numbers, and place the results directly over and below those crossed out, as $\frac{\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}}$. Then we proceed in this manner until we find that we have no two numbers that can be factored.

$$\frac{\overset{3}{\cancel{6}} \times 8 \times 12 \times \overset{3}{\cancel{18}} \times \overset{1}{\cancel{24}}}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{6}}} = ?$$

We then multiply together all the numbers above the line and use this as a dividend; and multiply together those below the line which gives us our divisor.

$$\frac{3 \times 8 \times 12 \times 3 \times 1}{1 \times 1 \times 1 \times 1 \times 1} = \frac{864}{1} = 864$$

and we obtain the result 864.

In an example where the product above the line after cancelling is less than the product below the line, we allow the result to stand as obtained; thus,

$$\frac{\overset{1}{\cancel{10}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{32}} \times \underset{3}{\cancel{6}} \times \underset{2}{\cancel{8}} \times 2} = \frac{1}{8} \text{ Ans.}$$

Example. Divide 18×5 by 10×3.

$$\frac{\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}} \quad \frac{\overset{3}{\cancel{18}} \times 5}{\underset{1}{\cancel{10}} \times \underset{1}{\cancel{3}}} = \frac{3}{1} = 3$$

We here see that 18 and 3 are both divisible by 3, so we can cancel them thus:

$$\frac{\overset{6}{\cancel{18}} \times 5}{10 \times \underset{1}{\cancel{3}}}$$

Placing a 6 in place of the 18 and a 1 in place of the 3. Similarly

5 is contained in 5 and in 10, so we replace the 5 with a 1 and the 10 with a 2. The 6 above the 18 and the 2 below the 10 now cancel and give the result $\frac{3}{1}$, which equals 3. These operations result in the following:

$$\frac{\overset{3}{\cancel{6}} \quad 1}{\cancel{18} \times \cancel{5}} \cdot \\ \cancel{10} \times \cancel{3} \\ \cancel{2} \quad 1 \\ 1$$

If we multiplied the numbers as they stood we should reach the same result thus:

$$\frac{18 \times 5}{10 \times 3} = \frac{90}{30} = 3$$

The following is another illustration. Divide $7 \times 4 \times 11 \times 16 \times 3$ by $22 \times 4 \times 4 \times 21$.

$$\frac{1 \quad 1 \quad 1 \quad \overset{2}{\cancel{4}} \quad 1}{\cancel{7} \times \cancel{4} \times \cancel{11} \times \cancel{16} \times \cancel{3}} = \frac{2}{1} = 2 \\ \cancel{22} \times \cancel{4} \times \quad \cancel{4} \times \cancel{21} \\ \cancel{2} \quad 1 \quad 1 \quad \cancel{3} \\ 1 \qquad\qquad 1$$

The cancellation is as follows:

The second figure (4) of each expression cancels, leaving a 1 in place of each.

The 16 of the dividend and 4 of the divisor cancel, leaving 4 and 1 respectively.

The 11 of the dividend and 22 of the divisor cancel, leaving 1 and 2 respectively.

The 7 of the dividend and 21 of the divisor cancel, leaving 1 and 3.

The 3 of the dividend and the 3 of the divisor under 21, cancel, giving 1 and 1.

The 4 over the 16 and the 2 under the 22 cancel, leaving 2 and 1.

This leaves the dividend $1 \times 1 \times 1 \times 2 \times 1$, and the divisor $1 \times 1 \times 1 \times 1$. Hence the answer is 2.

61. RULES. *1.* Cancel the common factors from both dividend and divisor.

2. Next divide product of the remaining factors of the dividend by the product of the remaining factors of the divisor.

Cancellation may be called a method of shortening arithmetical operations by rejecting factors common to the dividend and divisor.

EXAMPLES FOR PRACTICE.

(1) A grocer sold 24 boxes of soap, each containing 55 pounds, at 10 cents a pound, and received as pay 88 barrels of apples, each containing 3 bushels. How much were the apples worth a bushel? Ans. $.50 a bushel.

(2) A merchant bought 6 loads of oats, each load containing 22 bags, and each bag 2 bushels, worth 56 cents a bushel. He gave in payment 8 boxes of tea, each containing 24 pounds. What was the tea worth a pound? Ans. $.77 a pound.

(3) How many days' work, at $1.25 a day, will pay for 75 bushels of corn at $.80 a bushel? Ans. 48 days.

(4) Find the quotient of $\frac{12 \times 60 \times 27 \times 49}{7 \times 15 \times 42 \times 108}$. Ans. 2.

(5) Find the quotient of $\frac{77 \times 100 \times 18 \times 14}{25 \times 11 \times 49 \times 16}$. Ans. 9.

(6) Find the quotient of $\frac{16 \times 12 \times 7 \times 11}{24 \times 22 \times 7 \times 18}$. Ans. $\frac{2}{3}$.

(7) Find the quotient of $\frac{18 \times 100 \times 13 \times 12}{26 \times 25 \times 9 \times 3}$. Ans. 16.

Divide

(8) $2 \times 3 \times 8 \times 12 \times 24$ by $6 \times 4 \times 36 \times 4$. Ans. 4.

(9) $18 \times 24 \times 32 \times 36$ by $9 \times 48 \times 4 \times 18$. Ans. 16.

(10) $15 \times 20 \times 25 \times 27$ by $10 \times 15 \times 18 \times 25$. Ans. 3.

(11) $40 \times 48 \times 54 \times 60$ by $30 \times 24 \times 72 \times 3$. Ans. 40.

(12) $12 \times 60 \times 36 \times 70$ by $28 \times 5 \times 48 \times 6$. Ans. 45.

(13) $32 \times 36 \times 33 \times 45$ by $24 \times 30 \times 44 \times 9$. Ans. 6.

(14) $30 \times 36 \times 24 \times 42$ by $45 \times 27 \times 8 \times 28$. Ans. 4.

(15) $27 \times 32 \times 45 \times 36$ by $18 \times 24 \times 9 \times 6$. Ans. 60.

SECTION 2.

ARITHMETIC.

PART II.

FRACTIONS.

1. A part of a whole number is called a **Fraction.**

NOTE. A fraction may be regarded as expressing unexecuted division.

2. In expressing a fraction we use two numbers. For example, a foot is made up of twelve inches. Therefore, one inch is one-twelfth of a foot. We express it in figures $\frac{1}{12}$.

3. The part of the fraction above the line signifying division, is called the **Numerator;** the number below the line, the **Denominator.** The denominator shows *into how many parts* the unit is divided. The numerator shows *how many of these equal parts* are taken. Thus in the fraction $\frac{1}{12}$, the 12 shows that the number was divided into 12 parts, and the 1 shows that one of the twelve parts is taken. For example, $\frac{1}{12}$ of a foot would mean one inch, $\frac{2}{12}$ two inches, $\frac{3}{12}$ three inches, etc. The fraction $\frac{3}{5}$ designates 3 of 5 equal parts into which the unit is divided.

4. The value of a fraction depends on both the numerator and denominator. The *greater* the numerator the *greater* the value, the *greater* the denominator the *less* the value. The value is the numerator divided by the denominator. The line between the two numbers means *divided by.* If the numerator equals the denominator the value of the fraction is one. For example, $\frac{3}{3} = \frac{7}{7} = \frac{5}{5} = \frac{278}{278} = \frac{4}{4} = \frac{4156}{4156}$.

5. If the numerator of a fraction is less than the denominator it is called a **Proper Fraction,** as $\frac{10}{11}$. If the numerator exceeds the denominator it is called **Improper,** as $\frac{11}{10}$. A **Mixed Number** is a whole number (integer) and a fraction, as $7\frac{3}{8}$, which is equal to $7 + \frac{3}{8}$, and is read seven and three-eighths.

REDUCTION OF FRACTIONS.

6. The process of changing fractions in form without changing their value is called **Reduction of Fractions.**

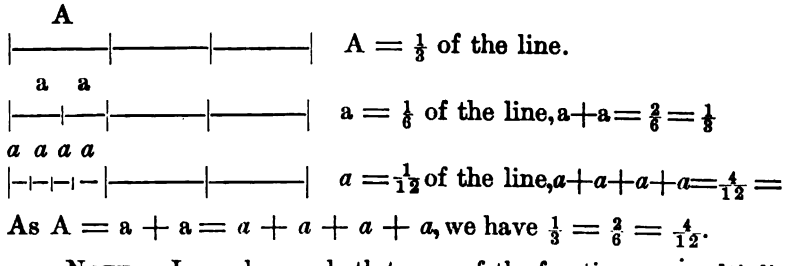
7. A fraction is reduced to higher terms by multiplying both terms of the fraction by the same number. For instance, multiply both terms of the fraction $\frac{1}{2}$ by 2.

$$\frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

We have not increased the value at all because $\frac{2}{2} = 1$ and multiplying (or dividing) by 1 does not change the value. $\frac{1}{2} = \frac{2}{4}$ just as half a dollar is equivalent to two quarters, the difference being simply a change in form.

NOTE. We could cancel $\frac{2}{4}$ in this way $\frac{1}{\cancel{2}\,\cancel{4}}$ and the result is $\frac{1}{2}$.

This is further illustrated thus:



As $A = a + a = a + a + a + a$, we have $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$.

NOTE. In each case both terms of the fraction are multiplied by 2 to produce the succeeding fraction.

8. A fraction is reduced to lower terms by dividing both numerator and denominator by the same number, thus

$$\frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

Dividing both terms of the fraction by the same number is equivalent to dividing the fraction by unity, and does not change the value.

A fraction is reduced to its lowest terms when its numerator and denominator cannot be divided by the same number without a remainder, for example $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{18}{19}$, $\frac{27}{82}$.

9. By the preceding paragraphs we see that we may change the terms of the fractions without changing their value as long as we perform the same operation on *both* terms of the fraction. For instance, if we wish to change $\frac{1}{2}$ to $\frac{2}{4}$ we may do so by multiplying each term by 2, just as we may change half a dollar for two quarters. Reversing the operation we may change $\frac{2}{4}$ to $\frac{1}{2}$. Similarly $\frac{1}{2}$ equals $\frac{50}{100}$, just as half a dollar is equivalent to fifty cents or fifty hundredths of a dollar. In this way we see that the following quantities all equal each other as the denominator is in each case twice the numerator $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14}$ $\frac{8}{16} = \frac{9}{18} = \frac{10}{20}$. Similarly, the denominator might always be three times as great as the numerator, so that the value would in each case be one-third, thus: $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$, etc.

10. It is not necessary, as we have already seen, that the numerator should be less than the denominator, for we have the contrary in the case of the improper fraction. The same laws, naturally, hold good. For instance, the numerator might be five times the denominator, thus: $\frac{5}{1} = \frac{10}{2} = \frac{15}{3}$, or it might be six times the denominator: $\frac{6}{1} = \frac{12}{2} = \frac{18}{3}$, or it might be seven times the denominator: $\frac{7}{1} = \frac{14}{2} = \frac{21}{3}$, etc.

11. Now we see by these two preceding paragraphs that we can change a fraction to an equivalent fraction with another denominator. Example: Change three-sevenths to an equivalent number of fourteenths, $\frac{3}{7}$ equals $\frac{?}{14}$. $14 \div 7 = 2$, hence $7 \times 2 = 14$, or to change the 7 to 14 we must multiply by 2. We wish to have the value of the fraction remain the same so we must multiply the 3 by the same number by which we multiply the 7, thus:

$$\frac{3 \times 2}{7 \times 2} = \frac{6}{14}$$

To illustrate further let us reduce $\frac{7}{8}$ to forty-eighths, 8 is contained in 48 six times, hence we have the following:

$$\frac{7 \times 6}{8 \times 6} = \frac{42}{48}$$

As another illustration reduce $\frac{3}{5}$ to hundredths. 5 is contained in 100 twenty times, so we have this result:

$$\frac{3 \times 20}{5 \times 20} = \frac{60}{100}$$

12. Now the reverse of this process may be performed by dividing each term of the fraction instead of multiplying. Thus to change $\frac{6}{14}$ to sevenths we divide each of the terms by 2 because 7 is contained twice in 14, thus:

$$\frac{6 \div 2}{14 \div 2} = \frac{3}{7}$$

Likewise, as 8 is contained six times in 48, we have—

$$\frac{42 \div 6}{48 \div 6} = \frac{7}{8}$$

Also, as 5 is contained twenty times in 100, we obtain the relation:

$$\frac{60 \div 20}{100 \div 20} = \frac{3}{5}$$

In short, we have reduced $\frac{42}{48}$ to eighths and $\frac{60}{100}$ to fifths.

13. In paragraph 8 we have shown how a fraction is reduced to lowest terms. When we see a fraction with both numerator and denominator divisible by the same number (other than one), we know that it can be reduced to lower terms. For example, in $\frac{6}{15}$ both numerator and denominator are divisible by 3, and by dividing each we have $\frac{2}{5}$ which is its lowest terms, because the 2 and 5 are no longer divisible. Likewise, if we have $\frac{10200}{34000}$ we see immediately that as each term ends in 0, we can divide each by 10 leaving $\frac{1020}{3400}$. Both 102 and 340 are divisible by 17; this gives us $\frac{6}{20}$, and as each of these terms is divisible by 2 we have $\frac{3}{10}$. Here we have to stop as 3 is a prime number, and can only be divided by itself and one, and 10 is not divisible by three without a remainder.

14. We may change mixed numbers to improper fractions. We may also write an integer (whole number) in fractional form. We may perform the reverse of these operations and change an improper fraction to a mixed number. The following examples illustrate these operations:

Change the mixed number $7\frac{5}{6}$ to an improper fraction.

$6 \times 7 = 42$ $\quad$ $42 + 5 = 47$ $\quad$ Ans. $\frac{47}{6}$

In this example we first take the integer 7 and see how many sixths it contains. As there are 6 sixths in each unit of the seven

units, we find that there are 42 sixths altogether. Now in the *fraction* of the mixed number there are 5 sixths. Adding these 5 sixths to the 42 sixths of the integer we find the whole number of sixths to be 47, so we write our result $\frac{47}{6}$.

Change $17\frac{12}{13}$ to an improper fraction.

$13 \times 17 = 221 =$ number of thirteenths in the integer (17).

$12 =$ number of thirteenths in fraction $(\frac{12}{13})$

$221 + 12 = 233 =$ number of thirteenths in mixed number $(17\frac{12}{13})$

Hence the answer is $\frac{233}{13}$.

NOTE. We see from the above that 17 might be written in the fractional form $\frac{221}{13}$. If we wish to write it as elevenths, we should have $\frac{187}{11}$, for $17 \times 11 = 187$.

If these fractions were reduced to lowest terms, we would obtain in each case 17 as a result. In performing the reverse of this process we proceed as follows:

Change $\frac{147}{16}$ to a mixed number.

$$16 \overline{)147} \quad 9\frac{3}{16}$$

This shows that in $\frac{147}{16}$ there are sufficient sixteenths to make 9 whole units with three sixteenths left over.

15. Rules for Fractions. To reduce a whole number or a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction (in the case of a whole number the denominator may be any number desired), and add to this the numerator of the fraction of the mixed number. Write this result as the numerator of the improper fraction, and place the denominator of the fraction of the mixed number below it as its denominator.

To change an improper fraction to a mixed number, divide its numerator by its denominator, write the whole number of the quotient as the integer of the mixed number, and the remainder over the divisor (denominator of improper fraction) as the fraction of the mixed number.

16. Before we can apply the two fundamental processes of addition and subtraction to fractions, we are obliged to write them with a common denominator, *i. e.*, the denominators must

be the same, as in the case of $\frac{7}{11}$ and $\frac{8}{11}$. The necessity of this will be demonstrated very shortly.

17. The **Least Common Multiple** of several numbers is the least number that will contain each of the several numbers without a remainder, *i. e.*, an integral number of times. The **Least Common Denominator** of several fractions is the least common multiple of their denominators, *i. e.*, the number which may be divided by each without leaving any remainder.

18. To illustrate how to find the least common multiple of several numbers, let us find that of 3, 4, 9, 12 and 16.

We first write these numbers in a row, separating them by commas, and proceed thus:

2	3,	4,	9,	12,	16
2	3,	2,	9,	6,	8
3	3,	1,	9,	3,	4
	1,	1,	3,	1,	4

$2 \times 2 \times 3 \times 3 \times 4 = 144$ Ans. (Least Common Multiple).

Now the least number to contain 3, 4, 9, 12, and 16, must be the smallest number that will contain the factors of each of them, but no other factors. We must, therefore, find all the prime factors that the multiple contains. 2 is a prime factor of 4, 12, and 16. Therefore it must be a factor of any number which contains them without a remainder. Dividing by 2 and writing the quotients below, wherever 2 is a factor, we have 3, 2, 9, 6, 8 (simply writing the numbers as they stand when 2 is not a factor). We see that 2 is a factor of 2, 6, and 8, so we divide by it in the same manner obtaining 3, 1, 9, 3, 4. Next we divide by 3. The result is 1, 1, 3, 1, 4. Now these have no common factor and must be factors of the least common multiple as 2, 2 and 3 are. Disregarding 1's we now have $2 \times 2 \times 3 \times 3 \times 4 = 144$. Least Common Multiple.

NOTE. We may disregard a 1 as soon as it appears in a row, as multiplying by it produces no change.

NOTE. We could find the least common multiple of 9, 12, and 16 and disregard 3 and 4 as both are contained without a remainder in 12, and any number which contains 12 without a remainder must contain 3 and 4 also.

To illustrate this note let us use 9, 12, and 16.

$$\begin{array}{r|lll} 2 & 9, & 12, & 16 \\ \hline 2 & 9, & 6, & 8 \\ \hline 3 & 9, & 3, & 4 \\ \hline & 3, & 1, & 4 \end{array}$$

$2 \times 2 \times 3 \times 3 \times 4 = 144$ which is the same result as before. Let us see if 144 is a common multiple.

$$9 \mid 144 \quad 16 \qquad 12 \mid 144 \quad 12 \qquad 16 \mid 144 \quad 9$$

Thus we see that 144 contains 9, 12, and 16 without a remainder.

19. In finding the least common denominator of a number of fractions, we simply find the least common multiple of their denominators. We change the numerators so that the fractions will have the same value as before they were written with a common denominator. For instance, let us write $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{1}{9}$ with a common denominator. The denominators are 3, 4, and 9. Let us find their least common multiple:

$$\begin{array}{r|lll} 3 & 3, & 4, & 9 \\ \hline & 1, & 4, & 3 \end{array}$$

By this we find the common denominator to be $3 \times 4 \times 3 = 36$.

Next we write these fractions with 36 as a denominator. As 3 is contained in 36 twelve times we have $\frac{1}{3} = \frac{12}{36}$

$$\frac{1 \times 12}{3 \times 12} = \frac{12}{36}$$

4 is contained in 36 nine times, hence $\frac{3}{4} = \frac{27}{36}$

$$\frac{3 \times 9}{4 \times 9} = \frac{27}{36}$$

9 is contained in 36 four times, hence $\frac{1}{9} = \frac{4}{36}$

$$\frac{1 \times 4}{9 \times 4} = \frac{4}{36}$$

Therefore we may write the fractions $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{1}{9}$ with the common denominator 36 as follows: $\frac{12}{36}$, $\frac{27}{36}$, $\frac{4}{36}$.

20. Rule for obtaining numerator of equivalent fraction when writing fractions with the common denominator of the fractions given: Multiply both terms of the given fraction by the quotient obtained by dividing common denominator by the denominator of the fraction.

EXAMPLES FOR PRACTICE.

Supply numbers in place of question marks.

1. $\frac{6}{11} = \frac{?}{44}$ Ans. $\frac{24}{44}$

2. $\frac{9}{28} = \frac{?}{56}$ Ans. $\frac{18}{56}$

3. $\frac{3}{7} = \frac{?}{84}$ Ans. $\frac{36}{84}$

4. $\frac{13}{19} = \frac{?}{57}$ Ans. $\frac{39}{57}$

5. $\frac{3}{51} = \frac{?}{153}$ Ans. $\frac{9}{153}$

6. $\frac{8}{22} = \frac{?}{88}$ Ans. $\frac{32}{88}$

Reduce to lowest terms.

7. $\frac{14}{84}$ Ans. $\frac{1}{6}$

8. $\frac{143}{429}$ Ans. $\frac{1}{3}$

9. $\frac{119}{476}$ Ans. $\frac{1}{4}$

10. $\frac{70}{350}$ Ans. $\frac{1}{5}$

11. $\frac{49}{63}$ Ans. $\frac{7}{9}$

12. $\frac{51}{68}$ Ans. $\frac{3}{4}$

Reduce to improper fractions.

13. $12\frac{3}{8}$ Ans. $\frac{99}{8}$

14. $18\frac{5}{9}$ Ans. $\frac{167}{9}$

15. $37\frac{18}{25}$ Ans. $\frac{943}{25}$

16. $87\frac{36}{41}$ Ans. $\frac{3603}{41}$

Reduce to whole or mixed numbers.

17. $\frac{395}{8}$ Ans. $49\frac{3}{8}$

18. $\frac{439}{34}$ Ans. $12\frac{31}{34}$

19. $\frac{143}{13}$ Ans. 11

20. $\frac{8947}{96}$ Ans. $93\frac{19}{96}$

Reduce to fractions with their least common denominator.

21. $\frac{2}{3}$ $\frac{3}{4}$ $\frac{5}{12}$ Ans. $\frac{8}{12}$ $\frac{9}{12}$ $\frac{5}{12}$

22. $\frac{5}{8}$ $\frac{6}{12}$ $\frac{3}{4}$ Ans. $\frac{15}{24}$ $\frac{12}{24}$ $\frac{18}{24}$

23. $\frac{4}{7}$ $\frac{3}{21}$ $\frac{13}{42}$ Ans. $\frac{24}{42}$ $\frac{6}{42}$ $\frac{13}{42}$

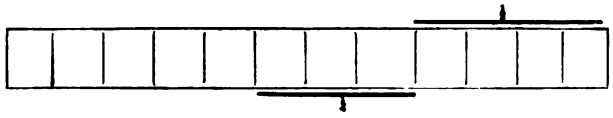
24. $\frac{5}{6}$ $\frac{4}{9}$ $\frac{7}{15}$ Ans. $\frac{75}{90}$ $\frac{40}{90}$ $\frac{42}{90}$

25. $\frac{57}{84}$ $\frac{67}{168}$ $\frac{59}{42}$ $\frac{3}{21}$ Ans. $\frac{114}{168}$ $\frac{67}{168}$ $\frac{236}{168}$ $\frac{24}{168}$

ADDITION OF FRACTIONS.

21. In adding fractions we simply reduce them to a common denominator and then add their numerators.

22. It is necessary to have a clear idea of why this is done. Let us observe the following illustration:



Suppose that we have a foot rule. What part of the whole rule is $\frac{1}{3}$ of a foot plus $\frac{1}{4}$ of a foot? We cannot write $\frac{1}{3} + \frac{1}{4}$ and express directly what part of the whole foot we have. The common denominator is 3×4 or 12.

$\frac{1}{3}$ of a foot is equal to $\frac{4}{12}$, as shown above.

$\frac{1}{4}$ of a foot is equal to $\frac{3}{12}$, as is also shown.

NOTE. Three times $\frac{1}{3}$ or three times $\frac{4}{12}$ make one foot. Similarly, $\frac{1}{4}$ or $\frac{3}{12}$ might be taken four times and the result be one foot.

From the above we see that $\frac{4}{12}$ plus $\frac{3}{12}$ equal $\frac{7}{12}$ or, since $\frac{1}{12}$ of a foot is equal to one inch, 7 inches. We could, therefore, write our result thus: $\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{4+3}{12} = \frac{7}{12}$ or simply $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ as the operation could be performed mentally. If there are more than two fractions the procedure is the same.

23. If we have mixed numbers, we first add the fractions and if the result of this addition is an improper fraction we change it to a mixed number, and add the integer in this mixed number to the sum of the integers of the mixed numbers to be added. Before adding the fractions they must, of course, be reduced to a common denominator.

For example add the following mixed numbers:

$11\frac{3}{4}$, $172\frac{5}{8}$ and $611\frac{11}{12}$

$$\begin{aligned} 11\tfrac{3}{4} &= 11\tfrac{18}{24} \\ 172\tfrac{5}{8} &= 172\tfrac{15}{24} \\ 611\tfrac{11}{12} &= 611\tfrac{22}{24} \\ \hline & 794 + \tfrac{55}{24} = 794 + 2\tfrac{7}{24} = 796\tfrac{7}{24} \text{ Ans.} \end{aligned}$$

24. RULES. *1.* Reduce the fractions to a common denominator.

2. Add their numerators and write the sum over the common denominator.

3. If there are mixed numbers and whole numbers, add the fractions first, and, if the sum is an improper fraction, change it to a mixed number and add the whole number or integer of this mixed number to the whole numbers to be added.

EXAMPLES FOR PRACTICE.

Find the sum of

(1) $\frac{5}{8}$, $\frac{7}{12}$ and $\frac{11}{24}$ Ans. $1\frac{3}{8}$

(2) $\frac{5}{16}$, $\frac{3}{20}$ and $\frac{4}{5}$ Ans. $1\frac{31}{80}$

(3) $\frac{3}{20}$, $\frac{7}{12}$ and $\frac{5}{6}$ Ans. $1\frac{17}{30}$

(4) $6\frac{2}{3}$, $2\frac{3}{7}$ and $4\frac{4}{15}$ Ans. $13\frac{23}{105}$

(5) $12\frac{7}{8}$, $8\frac{3}{5}$ and $2\frac{3}{20}$ Ans. $23\frac{17}{40}$

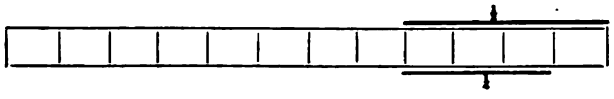
(6) $9\frac{4}{15}$, $\frac{7}{9}$ and $11\frac{13}{24}$ Ans. $21\frac{311}{360}$

(7) A room is $32\frac{3}{8}$ feet long and $29\frac{7}{8}$ feet wide. What is the distance around the room. Ans. $125\frac{1}{2}$ feet.

(8) Three castings weigh respectively $225\frac{3}{8}$, $232\frac{5}{12}$ and $240\frac{17}{20}$ pounds. What is their total weight? Ans. $698\frac{77}{120}$ lbs.

SUBTRACTION OF FRACTIONS.

25. Here we must reduce to a common denominator also. It is illustrated as follows:



By observing the above figure (which represents a foot rule, as before) we see that the difference between $\frac{1}{3}$ and $\frac{1}{4}$ is $\frac{1}{12}$ and that it is obtained by reducing to the common denominator, 12, and then subtracting the numerators instead of adding them. This is written thus:

$$\frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

To illustrate further take $\frac{3}{7}$ from $\frac{11}{14}$.

$$\frac{3}{7} = \frac{6}{14} \qquad \frac{11}{14} - \frac{6}{14} = \frac{7}{14}$$

Reducing $\frac{7}{14}$ to lowest terms we find by dividing both numerator and denominator by 7, that it is equal to $\frac{1}{2}$.

26. If we wish to subtract a fraction or mixed number from a whole number or from a mixed number in which the fraction of the minuend is less than the fraction of the subtrahend, we must write a part of the integer as a fraction. This is illustrated by the following examples:

From 17 take $\frac{9}{11}$.

We first write one of the 17 units as a fraction. As we know the number of elevenths in a unit is eleven, we may write the example in the following form:

$$\begin{array}{r} 16\frac{11}{11} \\ \frac{9}{11} \\ \hline 16\frac{2}{11} \end{array}$$

Subtracting $\frac{9}{11}$ from $\frac{11}{11}$ we have $\frac{2}{11}$, and, as we have nothing to subtract from 16, our answer is $16\frac{2}{11}$.

From $21\frac{3}{28}$ take $17\frac{9}{14}$.

We reduce to a common denominator. This gives us $21\frac{3}{28}$ and $17\frac{18}{28}$. $\frac{18}{28}$ is larger than $\frac{3}{28}$ so (since $\frac{28}{28} + \frac{3}{28} = \frac{31}{28}$) we write the example as follows:

$$\begin{array}{r} 20\frac{31}{28} \\ 17\frac{18}{28} \\ \hline 3\frac{13}{28} \end{array}$$

NOTE. If the fraction of the minuend exceeds that of the subtrahend, there is of course no necessity of increasing it.

27. RULES. *1.* Reduce the fractions to a common denominator.

2. Subtract the numerator of the subtrahend from that of the minuend and place the result over the common denominator.

3. If there are mixed numbers subtract the whole numbers and fractions separately, placing the fractional remainder after that of the whole number. If the fraction of the minuend is less than that of the subtrahend increase it by writing one of the units of the whole number in fractional form with it as an improper fraction. Do the same if there is no fraction in the minuend.

EXAMPLES FOR PRACTICE.

From

(1)	$\frac{12}{16}$	Subtract	$\frac{5}{8}$	Ans.	$\frac{1}{8}$
(2)	$\frac{5}{7}$	Subtract	$\frac{3}{5}$	Ans.	$\frac{4}{35}$
(3)	$\frac{5}{16}$	Subtract	$\frac{3}{24}$	Ans.	$\frac{3}{16}$
(4)	$\frac{19}{35}$	Subtract	$\frac{8}{45}$	Ans.	$\frac{23}{63}$
(5)	$\frac{12}{21}$	Subtract	$\frac{6}{49}$	Ans.	$\frac{22}{49}$
(6)	$\frac{17}{18}$	Subtract	$\frac{11}{27}$	Ans.	$\frac{29}{54}$

(7) $6\frac{3}{5}$ Subtract $2\frac{3}{7}$ Ans. $4\frac{6}{35}$

(8) $8\frac{5}{48}$ Subtract $2\frac{5}{24}$ Ans. $5\frac{43}{48}$

(9) $63\frac{4}{15}$ Subtract $39\frac{31}{35}$ Ans. $23\frac{8}{21}$

(10) A flag staff $50\frac{3}{10}$ feet high was broken off in a storm, so that it measured $43\frac{7}{8}$ feet. How much was broken off?

Ans. $6\frac{17}{40}$ feet.

(11) A box and its contents weighed $75\frac{3}{8}$ pounds. The box alone weighed $3\frac{5}{12}$ pounds. What was the weight of its contents? Ans. $71\frac{23}{24}$ lbs.

MULTIPLICATION OF FRACTIONS.

28. It is not necessary to reduce fractions to a common denominator when we multiply them.

29. We know from paragraph 4 that when we increase the numerator we increase the fraction, and that when we decrease the denominator we increase the fraction. Now we can multiply fractions in two ways. Usually we *multiply* the numerator. The same effect is produced by *dividing* the denominator by the same number. Let us take an example of this:

$$\frac{7}{8} \times 4 = \frac{28}{8} = \frac{7}{2} = 3\tfrac{1}{2}$$

(by multiplying numerator and reducing to mixed number).

$$\frac{7}{8} \div 4 = \frac{7}{2} = 3\tfrac{1}{2}$$

(by dividing denominator by 4 and reducing to mixed number).

30. If we say "of" in multiplication of fractions, we mean times ($\times$) just as we say three-quarters of a dollar. We could say three-fourths of a dollar just as we could say three-fourths of anything else. This is the same as multiplying by three-fourths, as $1 \times \frac{3}{4} = \frac{3}{4}$, and if the 1 were one dollar, the three-fourths would represent three quarters of a dollar.

31. If we multiply one fraction by another, we multiply the numerator or divide the denominator of the multiplicand by the numerator of the multiplier and perform the reverse operation with the denominator of the multiplier. This will be seen by the following examples:

Multiply $\frac{5}{8}$ by $\frac{4}{7}$.

The 4 of the multiplier makes the product larger. We must, therefore, either multiply the numerator or divide the denominator of the multiplicand by it. The 7 of the multiplier makes the product smaller. We must, therefore, either multiply the denomin-

ator or divide the numerator by it. Let us in this case take the usual course, and multiply the two numerators, and multiply also the two denominators.

$$\frac{5}{8}\times\frac{4}{7}=\frac{5\times 4}{8\times 7}=\frac{20}{56}=\frac{5}{14}$$

We may solve the problem by cancellation, as follows:

$$\frac{5}{8}\times\frac{4}{7}=\frac{5\times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}}\times 7}=\frac{5}{14}$$

32. In multiplying several fractions together it is the easiest method to use cancellation.

$$\frac{3}{11}\times\frac{22}{7}\times\frac{1}{18}\times\frac{9}{10}=\frac{3\times \overset{1}{\overset{2}{\cancel{22}}}\times 1\times \cancel{9}}{11\times 7\times \underset{1}{\overset{2}{\cancel{18}}}\times 10}=\frac{3}{70}$$

The cancellation is as follows:

The 11 in the denominator cancels with the 22 of the numerator leaving 2 in the numerator.

The 9 of the numerator and 18 of the denominator cancel, leaving 2 in the denominator.

The 2 over the 22, and the 2 under the 18 cancel.

Multiplying the remaining figures of the numerator we get 3.

Those of the denominator give 70. Hence our result is $\frac{3}{70}$.

33. If we wish to get the product of mixed numbers, we ordinarily change them to improper fractions and multiply.

For example. Multiply $2\frac{3}{7}$ by $5\frac{1}{4}$.

Writing these as improper fractions we have $\frac{17}{7}$ and $\frac{21}{4}$.

$$\frac{\overset{4}{\cancel{17}}}{\underset{1}{\cancel{7}}}\times\frac{\overset{3}{\cancel{21}}}{\underset{1}{\cancel{4}}}=\frac{12}{1}=12$$

We obtain this by precisely the same process as in the preceding examples.

34. If we have an integer and mixed number we multiply them as they stand.

$17\frac{3}{5}$
6

$103\frac{3}{5}=103\frac{1}{2}$

In detail

$17\frac{3}{5}$
6

$102+\frac{18}{5}=102+1\frac{3}{5}=103\frac{3}{5}=103\frac{1}{2}$

35. RULES. *1.* Divide product of the numerator by product of denominator. It is best first to cancel out all factors common to both numerator and denominator.

2. In multiplying mixed numbers first reduce them to improper fractions.

3. In multiplying a mixed number and an integer, first multiply fraction of mixed number by the integer (multiplier), reducing it, if it is an improper fraction, to a mixed number. Then multiply the integer of the mixed number by the integer. Add these two products.

EXAMPLES FOR PRACTICE.

Find the product of

(1) $\frac{5}{8}$ and $\frac{4}{7}$ Ans. $\frac{5}{14}$

(2) $\frac{8}{9}$ and $\frac{36}{37}$ Ans. $\frac{32}{37}$

(3) $\frac{12}{17}$ and $\frac{10}{13}$ Ans. $\frac{120}{221}$

(4) $7\frac{1}{8}$ and $8\frac{3}{7}$ Ans. $60\frac{3}{56}$

(5) $12\frac{3}{4}$ and $11\frac{5}{8}$ Ans. $147\frac{1}{8}$

(6) $11\frac{2}{7}$ and $8\frac{4}{5}$ Ans. $99\frac{11}{35}$

(7) $12\frac{3}{5}$ and 7 Ans. $88\frac{1}{5}$

(8) $11\frac{6}{7}$ and 8 Ans. $94\frac{6}{7}$

(9) $6\frac{5}{6}$ and $\frac{4}{5}$ and $4\frac{4}{5}$ Ans. $26\frac{6}{25}$

(10) $\frac{3}{5}$ and $\frac{4}{9}$ and $\frac{10}{13}$ and $\frac{12}{21}$ Ans. $\frac{32}{273}$

(11) There are $16\frac{1}{2}$ feet in a rod. How many are there in $\frac{7}{8}$ of a rod? Ans. $14\frac{7}{16}$ ft.

(12) A water-tube boiler has a grate surface of $27\frac{9}{10}$ square feet. It burns $27\frac{3}{4}$ pounds per square foot per hour. How much coal does it burn in $1\frac{1}{2}$ hours? Ans. $1161\frac{3}{8}\frac{7}{0}$ lbs.

(13) What will 142 yards of curbing cost at $\$6\frac{5}{8}$ per yard? Ans. $\$940\frac{3}{4}$.

(14) What will $17\frac{1}{4}$ tons of coal cost at $\$4\frac{7}{10}$ per ton? Ans. $\$82\frac{1}{2}$

DIVISION OF FRACTIONS.

36. In dividing fractions we simply reverse multiplication, and are not obliged to reduce to a common denominator.

37. As the process is the opposite of multiplication, *dividing* the numerator or *multiplying* the denominator divides the fraction.

For example divide $\frac{9}{10}$ by 3.

1. Dividing the numerator.

$$\frac{9 \div 3}{10} = \frac{3}{10}$$

2. Multiplying the denominator.

$$\frac{9}{10 \times 3} = \frac{9}{30} = \frac{3}{10}$$

NOTE. In the second case we reduce the quotient to lowest terms by dividing both numerator and denominator by 3. This gives us the same result as before.

38. Divide $\frac{27}{44}$ by $\frac{3}{4}$.

1. Dividing numerator by numerator and denominator by denominator.

$$\frac{27 \div 3}{44 \div 4} = \frac{9}{11}$$

2. Multiplying numerator by denominator, and denominator by numerator.

$$\frac{27 \times 4}{44 \times 3} = \frac{108}{132} = \frac{9}{11}$$

From the above we see that dividing any number by a fraction is the same as multiplying by the fraction inverted, *i.e.*, upside down.

39. In performing this process of multiplying with the divisor inverted, we may, of course, use cancellation.

For example divide $\frac{3}{17}$ by $\frac{7}{51}$.

$$\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{17}}} \times \frac{\overset{3}{\cancel{51}}}{7} = \frac{9}{7} = 1\frac{2}{7}$$

40. If we divide one by $\frac{1}{4}$ the result is 4. This is seen by the following:

$$1 \div \frac{1}{4} = 1 \times \frac{4}{1} = \frac{4}{1} = 4$$

We might write $1 \div \frac{1}{4} = 4$ in this fashion $\frac{1}{\frac{1}{4}} = 4$. From this we see that unity or one divided by a fraction inverts the fraction.

This is further seen by reasoning that as $\frac{1}{4}$ is contained 4 times in 1, the quotient of 1 divided by $\frac{1}{4}$ is 4 just as a quarter is $\frac{1}{4}$ of a dollar and it takes four quarters to make a dollar.

41. We treat mixed numbers in the same way.

For example:

How many times is $3\frac{3}{7}$ contained in $7\frac{3}{4}$ *i.e.*, what is the quotient of $7\frac{3}{4}$ divided by $3\frac{3}{7}$.

$$7\tfrac{3}{4} = \frac{31}{4} \qquad 3\tfrac{3}{7} = \frac{24}{7}$$

$$7\tfrac{3}{4} \div 3\tfrac{3}{7} = \frac{31}{4} \div \frac{24}{7} = \frac{31}{4} \times \frac{7}{24} = \frac{217}{96} = 2\tfrac{25}{96}$$

42. In dividing an integer by a fraction or mixed number the process is still easier. Divide 25 by $4\frac{3}{5}$.

$4\frac{3}{5}$ we write as the improper fraction $\frac{23}{5}$.

$$25 \div \frac{23}{5} = 25 \times \frac{5}{23} = \frac{125}{23} = 5\tfrac{10}{23}$$

43. RULE. Invert the divisor and then proceed in the same manner as with multiplication.

EXAMPLES FOR PRACTICE.

Divide

(1) $\frac{7}{9}$ by $\frac{4}{7}$ Ans. $1\frac{13}{36}$

(2) $\frac{7}{8}$ by $\frac{1}{4}$ Ans. $3\frac{1}{2}$

(3) $\frac{9}{13}$ by $\frac{3}{26}$ Ans. 6

(4) $\frac{19}{20}$ by $\frac{7}{20}$ Ans. $2\frac{5}{7}$

(5) $7\frac{3}{8}$ by $4\frac{1}{2}$ Ans. $1\frac{23}{36}$

(6) $11\frac{1}{4}$ by $5\frac{3}{7}$ Ans. $2\frac{11}{152}$

(7) $81\frac{1}{7}$ by $9\frac{1}{6}$ Ans. $8\frac{132}{161}$

(8) $17\frac{3}{8}$ by 6 Ans. $2\frac{43}{48}$

(9) $27\frac{11}{12}$ by 9 Ans. $3\frac{11}{108}$

(10) $189\frac{11}{15}$ by 4 Ans. $47\frac{13}{30}$

(11) How many times may $2\frac{1}{2}$ gallons be drawn from a barrel holding $31\frac{1}{2}$ gallons? Ans. $12\frac{3}{5}$

(12) A railroad $16\frac{1}{2}$ miles long cost $66,937. What did it cost per mile? Ans. $4,056\frac{26}{33}$

(13) A steam pump delivers $2\frac{3}{4}$ gallons per stroke. It delivers 330 gallons in $2\frac{1}{2}$ minutes. How many strokes does it make per minute? Ans. 48

44. We have seen that we could use a line in place of the symbol of division. If we wish to express such a relation between an integer and a fraction or a mixed number and another fraction, we make the line between them heavier than ordinarily. For instance, suppose we wish to write $11 \div \frac{3}{4}$. We could write thus, $\frac{11}{\frac{3}{4}}$. Let us now take the case of two fractions such as $\frac{11}{17}$ and $\frac{13}{41}$. We would write $\frac{11}{17} \div \frac{13}{41}$ in the following manner $\frac{\frac{11}{17}}{\frac{13}{41}}$ and as this is equal to the former expression we may write the following:

$$\frac{11}{17} \div \frac{13}{41} = \frac{\frac{11}{17}}{\frac{13}{41}} = \frac{11}{17} \times \frac{41}{13} = \frac{451}{221}$$

NOTE. $\frac{451}{221}$ will not reduce for the factors 11, 41, 17 and 13 are prime numbers and do not cancel.

45. Let us consider a fractional form of this kind a little further. Suppose the 11 to be increased. The result would be an increase in value of the whole expression. If we increase the 17 we would decrease the value of the expression. When, however, we write the $\frac{13}{41}$ after the sign $\times$ we see that it is inverted. Hence we perceive that if the 13 were replaced by a larger number, the value of the expression would be decreased. The opposite is true of the 41.

We may consider $\frac{11}{17}$ to be a numerator which is a fraction itself, and $\frac{13}{41}$ to be a denominator of the same nature. We have seen that the greater the value of the numerator the greater the value of a fraction. We know the opposite to be true of the denominator. This further shows us the relation of these numbers to each other.

We may make other combinations of signs, such as the following examples illustrate.

$$\frac{\frac{7+11}{5}}{\frac{14-2}{15}} = \frac{\frac{18}{5}}{\frac{12}{15}} = \frac{\cancel{3}}{\cancel{5}} \times \frac{\cancel{15}^{3}}{\cancel{12}_{2}} = \frac{9}{2} = 4\frac{1}{2}$$

If we have such an expression as $\frac{\cancel{8}}{\cancel{17}_{9}}$ we may cancel the 17 in the denominator of each term at once and write the result $\frac{8}{9} = \frac{1}{3}$.

DECIMALS.

46. We have seen that a unit of any order of units in our system of notation is ten times as great as a unit of the next lower order. We might state the same fact by saying that a unit of any order was one-tenth of a unit of the next higher order. Now we started with a unit and went up, a unit of any order being ten times a unit of the next succeeding order. Now, we can also begin with the unit place, and go in the opposite direction making each order one-tenth of the preceding order. Fractions written in this way are called **Decimals.** We must of course show in some way where the unit is and where the fractional part begins. We accomplish this by means of a period. We call this period a **Decimal Point,** and it is placed directly after units, *i. e.*, preceding the decimal part of the number. Of course if we write a decimal fraction this point is placed to the left of the numerator (with ciphers prefixed if necessary.) The denominator which is signified by the number of decimal places, is, as we shall see, always some power of 10 as, for example, 10, 100, 1,000, etc.

47. The reading of a decimal number depends on the number of decimal places in it, *i. e.*, the number of figures to the right of the decimal point.

We use the decimal system in money. For instance, take the sum $13.74. We may consider this to be made up of a ten dollar bill (tens position), 3 one dollar bills (units place, dollars being considered units), 7 dimes or tenths of a dollar (tenths position), and 4 cents or hundredths of a dollar (hundredths position.) If we read this thirteen dollars and seventy-four cents, we mean thirteen dollars and seventy-four hundredths of a dollar. It is, of course, better to say seventy-four hundredths than seven tenths plus four hundredths, just as it is better to say seventy-four cents than to say seven dimes and four cents. We mean $\frac{74}{100}$ of a dollar. If we wrote 1.04 we would mean a dollar and four hundredths of a dollar. This shows the use of the cipher in decimals.

48. The following table illustrates the relation of decimals to integers.

Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.	Ten-Thousandths.	Hundred-Thousandths.	Millionths.	Ten-Millionths.	Hundred-Millionths.
7	4	1	9	.6	2	3	4	9	7	8	1

48. As we see in the above, one decimal place expresses tenths, two express hundredths, three express thousandths, and so on. For example, we read .6 as 6 tenths, .62 as 62 hundredths, .623 as 623 thousandths, *i. e.*, just as we would read an ordinary integer but with the designation of the decimal place of the last figure numerated.

The above is a mixed number as it contains an integer (7,419) and a fraction (the decimal .62349781). The decimal point does not affect the reading of the integer. We would read the entire number thus: seven thousand, four hundred nineteen and sixty-two million, three hundred forty-nine thousand, seven hundred eighty-one hundred-millionths.

We see also in the same table that the same principle holds good throughout, that ten units of any order make one unit of the next higher order.

50. It is stated in what precedes that the denominator is always 10, 100, 1,000, etc., and that it is not expressed in figures. What the denominator actually is, we can tell by the

number of decimal places. This is shown by the following:

.7	$= \frac{7}{10}$	=	7 tenths.
.07	$= \frac{7}{100}$	=	7 hundredths.
.007	$= \frac{7}{1000}$	=	7 thousandths.
.0007	$= \frac{7}{10000}$	=	7 ten-thousandths.
.00007	$= \frac{7}{100000}$	=	7 hundred-thousandths.

Tenths
Hundredths
Thousandths
Ten Thousandths
Hundred Thousandths

We see that the place occupied by the last figure of the decimal gives its name to the fraction. We also see that the denominator of the equivalent ordinary fraction is simply 1 with as many zeros after it as there are places in the decimal. We also see that inserting a cipher between the decimal point and a number divides that number by ten. We must remember that the decimal point does not affect the value of figures when it is placed after them. We must also remember that it is the starting point of notation and numeration.

51. Annexing or taking away a cipher at the right of a decimal does not affect the value of a decimal. This is made clear by the following:

$.3 = \frac{3}{10}$; $.30 = \frac{30}{100}$ but, $\frac{30}{100}$ (reduced to lowest terms) is $\frac{3}{10}$, that is $\frac{3}{10} = \frac{30}{100}$, and therefore $.3 = .30$. We might continue thus: — $.3 = .30 = .300 = .3000$, etc.

Taking away ciphers at the right of a decimal is simply the reverse, *i.e.* $.3000 = .300 = .30 = .3$ as we have just seen.

52. It is between the decimal point and the number that the cipher produces a change in value. Inserting between point and number divides by ten. Removing a cipher between decimal point and number does the reverse—multiplies by ten. These facts are shown by the following:

$.3 = \frac{3}{10}$; $.03 = \frac{3}{100}$; $\frac{3}{10} \div 10 = \frac{3}{100}$ therefore $.3 \div 10 = .03$.

53. We have seen that to change a decimal to an ordinary fraction we write the decimal number as a numerator, and 1 with as many ciphers as there are places in the decimal, as a denominator. For example:

$.973 = \frac{973}{1000}$, $36.7413 = 36\frac{7413}{10000}$, $.043 = \frac{43}{1000}$, $.40 = \frac{40}{100}$.

To change a common fraction to an equivalent decimal we annex ciphers to the numerator and then divide it by the denominator, and point off as many decimal places in the quotient as we have annexed ciphers. We may annex as many ciphers as we like. If there is a remainder we express it as a common fraction, or, having found the decimal to as many places as we wish, we simply write a + to show that there is a remainder. The following is an illustration:

Change $\frac{11}{18}$ to a decimal.

$$18\overline{)11.0000}\ .6111\tfrac{2}{18}$$

We may write our quotient .6111 +

NOTE. If we continued to annex ciphers and divide we should get only 1 in the quotient for each division. Since we continually divide 20 by 18 our quotient is always 1 with 2 remainder. Therefore we may write our quotient as .6 followed by 1 any number of times and the fraction $\frac{2}{18}$ or $\frac{1}{9}$, as $.611111\frac{1}{9}$.

NOTE. If we had as a result $.736\frac{11}{12}$ or a similar number it would be more natural to write .737—instead of 736 + because $\frac{11}{12}$ is greater than $\frac{1}{2}$.

EXAMPLES FOR PRACTICE.

Reduce the following to common fractions in lowest terms:

(1)	.75	Ans. $\frac{3}{4}$
(2)	.35	Ans. $\frac{7}{20}$
(3)	.125	Ans. $\frac{1}{8}$
(4)	.625	Ans. $\frac{5}{8}$
(5)	.0045	Ans. $\frac{9}{2000}$
(6)	.00875	Ans. $\frac{7}{800}$
(7)	.00375	Ans. $\frac{3}{800}$
(8)	.05375	Ans. $\frac{43}{800}$

Reduce to equivalent decimals the following fractions:

(1) $\frac{3}{16}$ Ans. .1875

(2) $\frac{1}{40}$ Ans. .025

(3) $\frac{5}{11}$ Ans. .45+

(4) $\frac{4}{625}$ Ans. .0064

(5) $\frac{1}{6}$ Ans. .166+

(6) $\frac{3}{80}$ Ans. .0375

ADDITION OF DECIMALS.

54. The addition of decimals is performed in the same manner as in the case of whole numbers except that the numbers are placed with the decimal points directly under each other. The reason for this is obvious, as units must be under units, etc., and tenths under tenths, hundredths under hundredths, etc. The figures of each column are added and the decimal point of the answer placed under the column of decimal points.

For example add the following columns:

1.89	.948	62.589
62.625	.7642	.875
3.939	.035	1.3759
68.454	1.7472	64.8399

What is the sum of 5.018, 171.16, 88.133, 1113.6, .00456, 14.178.

5.018
171.16
88.133
1113.6
.00456
14.178

1392.09356

55. RULE. Place the numbers so that the decimal points will be directly under each other. Add as in the whole numbers, placing the decimal point in sum directly under the column of decimal points.

EXAMPLES FOR PRACTICE.

Find the sum of the following:

(1)	8.75 + .369 + 75.212	Ans.	84.331
(2)	3.801 + 2.875 + 1.625	Ans.	8.301
(3)	.356 + 1.275 + 32.0325	Ans.	33.6635
(4)	28.48 + 17.578 + 411.0007	Ans.	457.0587
(5)	1.8035 + 3.00412 + 31.025	Ans.	35.83262

(6) 3.495 + 1.87 + 3.255071 Ans. 8.620071

(7) .2735 + 1.596 + 3.25 + 1167.1 Ans. 1172.2195

(8) 171.61111 + 16.7101 + .00007 + 71.0006 + 1.167895. Ans. 260.489775.

(9) .16711 + 1.766 + 76111.1 + 167.1 + .000007 + 1476.1. Ans. 77756.233117

(10) 151.01 + 611111.01 + 16.5 + 6.7 + 46.1 + 678.95. Ans. 612010.27

(11) A bicyclist travelled 61.875 miles in one day, 55.75 the next, and 45.375 the next. How far did he travel altogether? Ans. 163 miles.

(12) The constituents of 100 cubic feet of uncarburetted gas weigh as follows: carbon dioxide .429065, carbon monoxide 3.38954, hydrogen .289821, nitrogen .102173. What is the total weight per 100 cubic feet? Ans. 4.210599

(13) The grate surfaces of four boilers are 14.9, 13.6, 13.6, and 16.7. What is their total grate surface? Ans. 58.8

(14) Horse-powers lost by friction in a small power house were as follows: engine, air-pump, and boiler feed pump 9.22, dynamo and belt 2.12. Total loss? Ans. 11.34 horse power.

SUBTRACTION OF DECIMALS.

66. In subtraction as in addition, and for the same reason we place the numbers so that their decimal points are under each other, and subtract as in the case of whole numbers. If figures in subtrahend extend beyond those of minuend, we add ciphers to the latter. The following examples illustrate the process.

From .673 take .452

Minuend	.673
Subtrahend	.452
Difference	.221

From 74.876 take 49.054

74.876
49.054
25.822

From 81.35 take 11.678956

81.350000
11.678956
69.671044

NOTE. We are here obliged to add four ciphers to the minuend, but as the minuend is a decimal, this addition of ciphers produces no change in its value.

From 17 take .739

17.000
.739
16.261

NOTE. We place the decimal point after the integer, and then we may add as many ciphers as we wish.

57. RULE. Place subtrahend under minuend so that the decimal points will be directly under each other. Subtract as with whole numbers placing the decimal point of the answer directly under the other two. When the number of figures in the decimal part of the subtrahend exceeds the number in the decimal

part of the minuend, add a sufficient number of ciphers to the latter to allow subtraction.

EXAMPLES FOR PRACTICE.

From

(1)	25.38 take 14.05	Ans. 11.33
(2)	39.85 take 29.755	Ans. 10.095
(3)	72.189 take 35.976	Ans. 36.213
(4)	78.895 take 53.5987	Ans. 25.2963
(5)	21.12 take 12.31	Ans. 8.81
(6)	6.325 take 1.0345	Ans. 5.2905
(7)	6.45 take 2.3375	Ans. 4.1125
(8)	81.35 take 11.678956	Ans. 69.671044
(9)	1 take .876543	Ans. .123457
(10)	100 take 99.111176	Ans. .888824
(11)	100 take .001	Ans. 99.999

(12) From a cistern that contained 30.5 barrels of water, 25.75 barrels were drawn off. How much water remained in the cistern? Ans. 4.75 barrels.

(13) What is the inside diameter of a pipe which is 7.34 inches outside diameter, and is made of iron .743 inch thick? Ans. 5.854 in.

(14) The average pressure on one side of a piston is 40.5 pounds. The atmospheric pressure which it acts against is 14.7 pounds. What is the effective pressure? Ans. 25.8 pounds.

(15) A hundred pounds of coke was found to contain 5.79 pounds of ash and .597 pounds of sulphur. The rest was fixed carbon. How much fixed carbon was there? Ans. 93.613 pounds.

MULTIPLICATION OF DECIMALS.

58. When we multiply with decimals we pay no attention to the decimal points until we have obtained all the figures of the answer. We then point off as many places in the answer (product) as there are in both multiplicand and multiplier.

We know that $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$. In the same way $.1 \times .1 = .01$, *i.e.*, the product contains as many decimal places as multiplicand and multiplier together.

For further illustration let us observe the following examples:

Multiply .397 by 41.

$$\begin{array}{r} .397 \\ 41 \\ \hline 397 \\ 1588 \\ \hline 16.277 \end{array}$$

NOTE. As the multiplier has no decimal places, we have in the product the same number as in the multiplicand.

Multiply 17.1 by .003.

$$\begin{array}{r} 17.1 \\ .003 \\ \hline .0513 \end{array}$$

NOTE. We need pay no attention to the ciphers of the multiplier as they merely determine the number of decimal places. We find one decimal place in the multiplicand and three in the multiplier. Therefore we have four (1 + 3) places in the product. We place a cipher in front of the figures we have found in order to allow us to point off four places.

Multiply 2.75 by .17.

$$\begin{array}{r} 2.75 \\ .17 \\ \hline 1925 \\ 275 \\ \hline .4675 \end{array}$$

Multiply 18.72 by 7.1.

$$\begin{array}{r} 18.72 \\ 7.1 \\ \hline 1872 \\ 13104 \\ \hline 132.912 \end{array}$$

Multiply 5.12 by .012.

$$\begin{array}{r} 5.12 \\ .012 \\ \hline 1024 \\ 512 \\ \hline .06144 \end{array}$$

Multiply .679 by .0001.

$$\begin{array}{r} .679 \\ .0001 \\ \hline .0000679 \end{array}$$

NOTE. As the multiplier contains three decimal places and the multiplicand four, the product has $3 + 4$ or seven places as is seen.

59. RULE. Place multiplier under multiplicand without paying attention to decimal points. Multiply as in whole numbers, and point off as many places in the product as there are in both multiplicand and multiplier, prefixing ciphers if necessary.

EXAMPLES FOR PRACTICE.

Find the product of

(1)	.876 and .375	Ans. .3285
(2)	72.2 and .055	Ans. 3.971
(3)	3.62 and .0037	Ans. .013394
(4)	15.8 and .0855	Ans. 1.3509
(5)	2.53 and .00635	Ans. .0160655
(6)	.765 and .067	Ans. .051255

(7) 18.46 and 1.007 Ans. 18.58922

(8) .00076 and .0015 Ans. .00000114

(9) .0701 and .0067 Ans. .00046967

(10) .47 and 47 Ans. 22.09

(11) A U. S. gallon of water weighs 8.355 pounds. What is the weight of 17.3 gallons? Ans. 144.5415 lbs.

(12) A steamship sailed at an average speed of 325.75 miles per day. If another steamer sailed from the same port at the same time at the rate of 395.35 miles per day, how far apart were they in 5.75 days? Ans. 400.2 miles

(13) A cubic foot of water weighs 62.5 pounds. How much will 177.35 cubic feet of water weigh? Ans. 11084.375 lbs.

(14) The stroke of an engine is 3.117 feet. It makes 107 revolutions per minute. Over what total distance will the cross-head travel in 4.5 minutes? Ans. 3001.671 ft.

NOTE. There are two strokes for each revolution.

DIVISION OF DECIMALS.

60. In division of decimals we pay no attention to the decimal point until after obtaining all the figures of the quotient. We first make sure that there are as many decimal places in the dividend as there are in the divisor. This we accomplish by annexing ciphers to the dividend until it has as many decimal places as the divisor, unless the dividend has as many as, or more decimal places than the divisor, in which case the operation is unnecessary. We then divide *exactly* as with whole numbers. Finally, *subtract the number of decimal places in the divisor* from *the number of decimal places in the dividend*, and point off the *resulting number of places* in our quotient. The following are illustrations of the process.

Divide 36.744 by 12.

```
12)36.744(3.062
   36
   ---
    74
    72
    --
     24
     24
```

NOTE. As there are no decimal places in the divisor and three in the dividend, we have nothing to subtract from three, so that the number of decimal places in the quotient is three.

Divide .196 by .004.

```
.004).196(49
    16
   ---
    36
    36
   ---
     0
```

NOTE. We have here the same number of decimal places in both dividend and divisor. For this reason there are no places to be pointed off in the quotient and hence it is a whole number.

Divide 794 by .25.

```
.25)794.00(3176
    75
   ---
    44
    25
   ---
   190
   175
   ---
    150
    150
   ---
      0
```

NOTE. We here add two ciphers to the dividend so as to have the same number of decimals in dividend and divisor. Then dividing we obtain a whole number.

Let us observe this example a little further. The divisor, .25, equals $\frac{25}{100}$ or $\frac{1}{4}$. If we divide by $\frac{1}{4}$ the result is the same as when we multiply by 4.

Multiplying 794 by 4 we obtain 3176.

Divide .0027 by 1.35.

```
1.35).00270(.002
     270
    ---
      0
```

NOTE. In dividing, as usual, we consider the figures as whole numbers. Now, 135 is not contained in 27. We have seen that to add a cipher does not change the value of a decimal. We add a cipher to the dividend. Then we have 270 as a dividend and the divisor 1.35 is contained twice. Since we have added a cipher to the dividend it has five decimal places. The divisor has two. Hence the quotient must have 5 — 2 or three decimal places. In order to have three decimal places we prefix two ciphers to the quotient 2 and place the decimal point in front of them.

We may use this method in dividing the smaller of two numbers by the larger.

Divide 35 by 125.

$$\begin{array}{r} 125)35.00(.28 \\ 250 \\ \hline 1000 \\ 1000 \\ \hline 0 \end{array}$$

NOTE. In order to divide 35 by 125 we have annexed two ciphers. We then get an answer and no remainder. By annexing the two ciphers we get two decimal places. Therefore, as there are two decimal places in the dividend and none in divisor we have two decimal places in the quotient (.28).

Let us perform the division another way.

$$\frac{35}{125} \div \frac{5}{5} = \frac{7}{25} \text{ and } \frac{7}{25} \times \frac{4}{4} = \frac{28}{100} \text{ or } .28$$

This is the same as

$$\frac{35}{125} \times \frac{\frac{1}{5}}{\frac{1}{5}} = \frac{28}{100}$$

61. We may use fractions in expressing a remainder, thus:

Divide 283 by 17.

$$\begin{array}{r} 17)283(16\frac{11}{17} \\ 17 \\ \hline 113 \\ 102 \\ \hline 11 \end{array}$$

The quotient is here $16\frac{11}{17}$.

$$
\begin{array}{r}
17)283.000(16.647 + \\
17 \\
\hline
113 \\
102 \\
\hline
110 \\
102 \\
\hline
80 \\
68 \\
\hline
120 \\
119 \\
\hline
1
\end{array}
$$

The quotient is here 16.647 +.

Let us change the fraction $\frac{11}{17}$ to a decimal.

$$
\begin{array}{r}
17)11.000(.647 + \\
102 \\
\hline
80 \\
68 \\
\hline
120 \\
119 \\
\hline
1
\end{array}
$$

This further illustrates the relations of the two methods.

$$16\tfrac{11}{17} = 16.647 +$$

Taking away each 16 we have

$$\tfrac{11}{17} = .647 +$$

RULE. Proceed as in division of whole numbers, but point off in the quotient as many decimal places as the number of decimal places in the dividend exceeds the number of decimal places in the divisor, prefixing ciphers to the quotient when necessary.

NOTE. In division it is usually best to carry on the division to three or four decimal places by annexing a sufficient number of ciphers to the dividend, and then, if there is a remainder, to signify the fact by the sign +.

EXAMPLES FOR PRACTICE.

Divide

(1)	183.375 by 489	Ans. .375
(2)	67.8632 by 32.8	Ans. 2.069
(3)	67.56785 by .035	Ans, 1930.51
(4)	.567891 by 8.2	Ans. .069255
(5)	.1728 by 100	Ans. .001728
(6)	13.50192 by 1.38	Ans. 9.784
(7)	783.5 by 6.25	Ans. 125.36
(8)	983 by 6.6	Ans. 148.93+
(9)	3 by 8	Ans. .375
(10)	1.935 by .45	Ans. 4.3

(11) How much gas at $2.50 per thousand cubic feet can be bought for $17.50? Ans. 7000 cubic feet.

(12) The distance between two places is 167.75 miles. How long will it take a steamer to run the distance, if she makes, on the average, 12.5 miles per hour? Ans. 13.42 hours.

(13) If a freight train runs at the rate of 15.75 miles per hour, how long will it take it to run 189 miles? Ans. 12 hours.

(14) A lot of 22,840 railroad ties cost $3,986,722. What was the cost per hundred? Ans. $17,455.

(15) Seven readings of a dynamometer gave the following horse-powers: 17.31, 17.95, 18.13, 17.79, 17.87, 17.63, and 17.46. What was the average reading? Ans. 17.734+.

SYMBOLS OF AGGREGATION.

62. These are the following: *parenthesis* (), *brackets* [], brace $\{\}$, and vinculum, $\overline{}$. They are used to include members which are to be taken *as a group* by themselves, standing at the same time in another expression. For example observe the following expressions:

$$12 \times 9 - 7$$

The value of this expression is 12×9 or 108 minus 7 or 101. Now if we put the $9 - 7$ in a *group* by itself, the value of the expression changes. Add the parenthesis thus:

$$12 \times (9 - 7)$$

Now we have *combined* the 9 and 7. This gives us the value of 2 for the parenthesis, because $9 - 7 = 2$ and $12 \times 2 = 24$, therefore we have

$$12 \times (9 - 7) = 24$$

63. If we have a series of numbers written in the manner just quoted, we perform the operations indicated by the signs ($+$, $-$, $\times$, $\div$) excepting in the following cases: When the sign of multiplication or division follows a number we do not add or subtract it *until* we have multiplied or divided it (by the following number or numbers) as the case may be. This may be summed up by saying that before finding the value of the expression we multiply or divide all numbers connected by the $\times$ and $\div$ signs.

The reason for this is that numbers connected by the multiplication sign are regarded as factors of a number (the one by which we replace them). Hence the multiplication signs take precedence. Division is multiplication reversed.

When numbers have a *symbol of aggregation* we find their value separately and use that value in place of the whole group.

Let us take a few examples of these figures:

What does $8 \times 17 - 11 + 13$ equal?

We first multiply the 8 and 17 obtaining 136, next we subtract 11, making the number 125, and then complete the process by adding 13 making the value of the expression 138.

What is the value of $4291 \div 7 + 141 - 24 \times 2\frac{1}{2}$?

$$4291 \div 7 = 613$$

$$613 + 141 = 754$$

Here we are obliged to multiply 24 and $2\frac{1}{2}$ before continuing because they are connected by the $\times$ sign.

$$24 \times 2\tfrac{1}{2} = 60$$
$$754 - 60 = 694$$

Hence $4291 \div 7 + 141 - 24 \times 2\frac{1}{2} = 694$.

What is the value of $36 \div 4 + 141 - (4 \times 11 \div 2)$?

$$36 \div 4 = 9$$
$$9 + 141 = 150$$

We cannot now proceed until we have multiplied 4 by 11 and divided the product by 2 because, as before stated, the $\times$ and $\div$ take precedence.

$$4 \times 11 = 44$$
$$44 \div 2 = 22$$

We may now subtract and get our total result.

$$150 - 22 = 128.$$

Therefore $36 \div 4 + 141 - (4 \times 11 \div 2) = 128$.

NOTE. The value of a symbol of aggregation within another is found before finding the value of the latter.

For example:

$$(\overline{17 + 3} \div 5) + 47.$$

$17 + 3 = 20$ value of expression under vinculum.
$20 \div 5 = 4$ value of total bracketed expression.
$4 + 47 = 51$.
Hence our answer is 51.

EXAMPLES FOR PRACTICE.

(1)	$(9 + 5) \div 2$	Ans. 7
(2)	$5 \times 18 - 25$	Ans. 65
(3)	$4 \times (17 + 4)$	Ans. 84
(4)	$(117 - 11) \div 2$	Ans. 53
(5)	$(120 - 75 + 45) \div 9$	Ans. 10
(6)	$\overline{13 + 7} \div \{3 + 2\}$	Ans. 4
(7)	$(43 + 2 - 11) \div (9 + 8)$	Ans. 2
(8)	$(48 + 7) \div 11 \times (5 + 9)$	Ans. 70

SECTION 3.

ARITHMETIC.

PART III.

PERCENTAGE.

1. **Percentage** and per cent are derived from Latin words per centum, which mean by the hundred.

Percentage is a method of calculating by hundredths.

2. We know that 20 is a certain part of 100; it is $\frac{20}{100}$. Instead of saying 20 is $\frac{20}{100}$ of 100 we say it is 20 per cent of 100. Then 20 per cent is the same as $\frac{20}{100}$, and decimally it is .20 since $\frac{20}{100} = .20$. There is still another way to write 20 per cent; it is 20%. We use the sign % in place of the words per cent. Thus 6 per cent is written 6% and 25 per cent, 25%. Thus we see that per cent may be written in various ways.

$$6 \text{ per cent} = \frac{6}{100} = .06 = 6\%.$$

The sign % is equivalent to two decimal places. For instance .03 is the same as 3%. In solving problems to find percentage we often get an answer containing a decimal; to reduce the decimal to the per cent form we move the decimal point two places to the right and annex the sign %.

For example: What per cent of 620 is 31?

$$\begin{array}{r} 620)\ 31.00\ (\ .05 \\ \underline{31\ 00} \\ 0 \end{array}$$

31 equals .05 of 620 or it is 5%.

We know from the study of decimals that .06 is six one hundredths, .15 is fifteen one hundredths. Now .06 is equal to $\frac{6}{100}$; hence we have only to point off two places to write 6 per cent as a decimal. 6 per cent may be expressed as a fraction whose denominator is 100 and we know that two decimal places stand for hundredths. Thus, 23 per cent may be written .23

3. Suppose we wish to find 6% of 560. We simply multiply $560 \times .06 = 33.6$. We multiply by .06 because 6% = .06. Similarly 5% of 800 is $800 \times .05 = 40$.

We know that $20\% = \frac{20}{100} = \frac{1}{5}$. If this simple fraction is used when multiplying to find per cent the process will be shortened.

We have seen how 20 per cent of a number is $\frac{20}{100}$ of it. Now we can reduce this fraction to lower terms as $\frac{20}{100} = \frac{1}{5}$. Similarly 50 per cent equals $\frac{50}{100} = \frac{1}{2}$. If we wish to find 20 per cent of a number we can multiply by $\frac{20}{100}$ or by .20 or find $\frac{1}{5}$ of it.

The following table illustrates the relation of per cent, decimals and fractions.

1 per cent	1%	.01	$\frac{1}{100}$
5 per cent	5%	.05	$\frac{5}{100} = \frac{1}{20}$
50 per cent	50%	.50	$\frac{50}{100} = \frac{1}{2}$
89 per cent	89%	.89	$\frac{89}{100}$
100 per cent	100%	1.00	$\frac{100}{100} = 1.$

If the per cent is less than 1 or a part of one hundredth it is expressed decimally by writing it to the right of the hundredths place. Thus $\frac{1}{2}$ of 1% is .00$\frac{1}{2}$ or .005. $\frac{3}{4}$ of 1% is .0075. 12$\frac{1}{2}$ per cent is .12$\frac{1}{2}$ or .125.

The following table shows this relation when the per cents are fractions of 1 per cent or whole numbers with fractions.

$\frac{1}{10}$%	.001	$\frac{\frac{1}{10}}{100} = \frac{1}{1000}$
$\frac{1}{4}$%	.0025	$\frac{\frac{1}{4}}{100} = \frac{1}{400}$
8$\frac{1}{3}$%	.08$\frac{1}{3}$	$\frac{8\frac{1}{3}}{100} = \frac{1}{12}$
37$\frac{1}{2}$%	.375	$\frac{37\frac{1}{2}}{100} = \frac{3}{8}$

4. We use several terms in percentage. They are as follows:

The **Base** is the whole number the percentage of which is taken.

The **Rate** is the *number of hundredths* or per cent of the base.

The **Percentage** is the number found by taking *a per cent or number of hundredths* of the base.

The **Amount** is the sum of the base and the percentage.

The **Remainder** is the part of the base left after subtracting the percentage.

In order to illustrate the preceding, let us take an example. Find four per cent of two hundred dollars.

The base is \$200.00 and the rate is $4\% = \frac{4}{100} = .04$.

The percentage is \$200.00 × .04 or \$8.00

The amount is \$200.00 + \$8.00 or \$208.00

The remainder is \$200.00 — \$8.00 or \$192.00

5. It is evident from the preceding that, to find the percentage, we multiply the base by the rate. It is best to express the rate as a decimal. Take the following example:

Of 150 armor plates 6% were condemned, how many were condemned?

Here 150 is the base and 6% (expressed decimally .06) is the rate. Hence, by multiplying, we have

$$\begin{array}{r} 150 \\ .06 \\ \hline 9.00 \end{array}$$

That is, 9 plates were condemned.

Of 200 boiler tubes 87% are used in the construction of a boiler. How many are used?

Here 200 is the base and 87% (expressed decimally .87) is the rate. Hence, multiplying, we have

$$\begin{array}{r} 200 \\ .87 \\ \hline 1400 \\ 1600 \\ \hline 174.00 \end{array}$$

That is, 174 tubes were used.

6. If we know the percentage and rate, and wish to find the base, we divide the percentage by the rate.

To show this more plainly let us take the following example, 15 is the percentage and 3% the rate. What is the base? Now we know that 15 is 3% or $\frac{3}{100}$, therefore $\frac{1}{100}$ is $\frac{1}{3}$ of 15 or 5, and $\frac{100}{100}$ or the base is 500. This operation is simply dividing 15 by $\frac{3}{100}$ (which is, as we know, the same as multiplying by $\frac{100}{3}$), that is, we have divided the 15 by 3 and multiplied it by 100 (or multiplied by 100 and divided by 3 which has the same effect). The usual method would be the following, $\frac{15}{.03} = 500$.

Let us take again the illustration of the boiler tubes, 87% of a quantity of boiler tubes are used. The number of tubes used

was 174. What was the number of tubes in the whole quantity, and how many were left after fitting the boiler?

$$174 \div .87 = 200$$

Since there were 200 to start with and 174 used, there were 26 left over. This 26 is 1.00 — .87 or 13 %

7. When the base and percentage are given, the rate is found by dividing the percentage by the base. We can, perhaps, perceive this fact the more readily if we remember that the percentage divided by the base expresses the fractional part of the base which the percentage is, and that the rate expresses the same thing in hundredths, or in other words, the rate shows what part of the base the percentage is. Take an example.

Of 200 boiler tubes 174 were used. What per cent of the total quantity was used?

$$174 \div 200 = .87$$

Hence 87 % was used.

8. The last two processes can be derived directly from the first in the following manner:

Percentage = *base* × *rate.*

By this we see that the number of times the rate is contained in the percentage is the base, and that the number of times the base is contained in the percentage is the rate, hence we see that *base* = *percentage* ÷ *rate*, and *rate* = *percentage* ÷ *base.*

9. Now we know that the amount is equal to the base plus the percentage. These might be added directly, but there is a shorter method of adding the same number or fractional parts of it, *i. e.*, multiplication. Therefore multiply the base by one plus the rate. For example:

A train is running at 25 miles per hour. The speed is increased 20 %. At what speed does the train then run?

$$25 \times (1 + .20) = 25 \times 1.20 = 30$$

Hence the speed of the train has increased to 30 miles.

We might, as first mentioned, obtain the result thus:

$$25 \times .20 = 5 \qquad 25 + 5 = 30$$

10. Now in the same way the difference might be found by subtracting the percentage (separately found) from the base.

It is easier, however, to multiply the base by one minus the rate.

Let us take a similar example:

The speed of a train running 25 miles an hour is decreased 20%. At what speed does it then run?

$$25 \times (1 - .20) = 25 \times .80 = 20$$

Hence 20 miles an hour is the decreased speed of the train.

Let us apply this to the example in boiler-tubes. Of 200 tubes 13 per cent. were not used in a boiler. The rest were. How many were used?

$$200 \times (1 - .13) = 200 \times .87 = 174. \quad \text{Ans.}$$

NOTE. It is well to observe that $200 \times .87$ has already been seen in paragraph 5.

11. Suppose that the amount and rate are given. We now have the amount which is equal to the base plus the percentage. The base is, of course, 100% or $\frac{100}{100}$. The rate is also given, and is, as we have already seen, equal—in hundredths—to itself over 100. Let us take an example:

515 is the amount. The rate is 3%. What is the base?

From this we see that 515 is equal to $\frac{100}{100} + \frac{3}{100}$ or $\frac{103}{100}$. If 515 is $\frac{103}{100}$ then $\frac{1}{100}$ must be 5 and $\frac{100}{100}$ or the base must be 500.

We see that, by the same method used in paragraph 6, we have simply divided 515 by $\frac{103}{100}$.

NOTE. Observing that $\frac{103}{100}$ is greater than unity, we see that our result must be less than 515 as it is (500).

Let us take another example:

A train is running at 30 miles per hour. This number is 20% greater than its usual speed. What is its usual speed?

Here $30 = \frac{100}{100} + \frac{20}{100} = \frac{120}{100}$. Therefore $\frac{100}{100} = 25$.

Or by the ordinary method, dividing by $1 + .20$.

$$\frac{30}{1.20} = 25 \text{ because } 1.20\overline{)30.00}\ \underline{25}$$

12. If the difference and rate are given the operation is practically the same except that we know the difference to equal a number of hundredths equal to 100 *less* the rate. To take an example:

485 is the difference, and 3% is the rate. What is the base?

Here we see that $485 = \frac{100}{100} - \frac{3}{100} = \frac{97}{100}$, and if 485 is $\frac{97}{100}$ then $\frac{100}{100}$ is 500 which is the base.

The usual procedure gives the same result $\frac{485}{.97} = 500$.

To take another example:

A train is running 20 miles per hour. This is 20% less than its usual speed. What is the usual speed?

$20 = \frac{100}{100} - \frac{20}{100} = \frac{80}{100}$. Therefore $\frac{100}{100} = \frac{20}{\frac{80}{100}} = 25$.

By the usual method: $20 \div (1 - .20) = 20 \div .80 = 25$.

NOTE. As the divisor (.80) is less than unity the quotient (25) must be greater than the dividend (20) as we see.

13. Suppose now we wish to find the per cent gained or lost. Let us again take the example of the train.

A train is running at the rate of 25 miles per hour. The speed is increased to 30 miles per hour. What is the gain per cent?

The number of miles gained or *percentage* is 30 — 25 or 5 and 5 is 20 per cent of 25 for $5 \div 25 = .20$. This means that the train runs 20% faster than before.

Suppose now, that the train was running at 25 miles per hour, but the speed has been decreased to 20 miles. What is the loss per cent?

The number of miles lost or percentage is 25 — 20 or 5. Now 5 is 20% of 25. Therefore the train runs at 20% less speed than before.

A man buys an engine for 1,250 dollars and a boiler for 90% of that price. If he sells the two for 2,200 dollars, does he gain or lose, and what per cent of what he paid?

The amount which he paid is 1,250 dollars and 90% of 1,250 dollars, which whole sum is $1{,}250 \times (1 + .90) = 2{,}375$ dollars.

He sold for 2,200 and therefore lost 2,375 — 2,200 or 175 dollars.

The per cent lost equals the percentage divided by the base, or $\frac{175}{2375}$ which is .0737 — or 7.37%.

14. RULES. To find the percentage, multiply the base by the rate.

To find the base when the percentage and rate are given, divide the percentage by the rate, expressed decimally.

To find the rate when the percentage and base are given divide the percentage by the base, and the quotient is the rate, expressed decimally.

To find the amount when the base and rate are given, multiply the base by one plus the rate, expressed decimally.

To find the difference when the base and rate are given, multiply the base by one minus the rate, expressed decimally.

To find the base when the amount and rate are given, divide the amount by one plus the rate, expressed decimally.

To find the base when the difference and rate are given, divide the difference by one minus the rate expressed decimally.

To find the per cent gained or lost, find the difference between the initial and final values, and divide this difference by the initial (first) value.

NOTE. These rules say "expressed decimally," simply because that is the most convenient method.

These rules may be summarized thus:

Percentage = Base × Rate.
Base = Percentage ÷ Rate.
Rate = Percentage ÷ Base.
Amount = Base × (1 + Rate).
Difference = Base × (1 — Rate).
Base = Amount ÷ (1 + Rate).
Base = Difference ÷ (1 — Rate).

Gain or loss per cent. = difference between initial and final values, ÷ initial value.

EXAMPLES FOR PRACTICE.

(1)	What is 12% of $940?	Ans. $112.80
(2)	What is 62% of 2000 men?	Ans. 1240 men.
(3)	What is $\frac{3}{5}$% of $28.80?	Ans. $.108
(4)	What is $37\frac{1}{2}$% of 4000 feet?	Ans. 1500 feet
(5)	300 is $7\frac{1}{2}$% of what number?	Ans. 4000
(6)	1199 is 88% of what number?	Ans. 1362.5
(7)	What per cent of $104 is $18.20?	Ans. $17\frac{1}{2}$%
(8)	What per cent of 300 is 272?	Ans. $90\frac{2}{3}$%

(9) What per cent of 12 is 20? Ans. $166\frac{2}{3}\%$.

(10) 2,352 pounds of coal are used during a 5 hour test. The weight of the resulting ashes is 56 pounds. What is the per cent of ashes? Ans. $2.38+\%$.

(11) The average number of revolutions of an engine is 140 per minute. The maximum speed is 15% more revolutions. What is the maximum number of revolutions? Ans. 161.

(12) The surface of the earth contains about 144 million square miles of water, and about 53 million square miles of land. What per cent of the entire surface of the earth is water?

Ans. $73+\%$.

(13) If a piece of rubber hose shrinks 4% when wet, what was its length when dry, if it is 192 feet long when wet?

Ans. 200 feet.

(14) From a cask containing 120 gallons of oil, $6\frac{1}{2}$ gallons leaked out. What per cent was lost? Ans. $5.4+\%$.

(15) A furnace consumes 114 bushels of coal per day. If it is improved so that 25% of the coal is saved, how many bushels are saved? Ans. $28\frac{1}{2}$ bushels.

(16) Ninety bushels of coal contain 5 bushels of slate. They produce 2 bushels of clinkers and 10 bushels of ashes. What is the per cent of each in the 90 bushels of coal? Ans. Slate $5\frac{5}{9}\%$, Clinkers $2\frac{2}{9}\%$, Ashes $11\frac{1}{9}\%$

(17) The tensile strength of a sheet of boiler plate is 270,000 pounds. If the joint of the boiler has only 69 per cent of this strength, how much has it? Ans. 186,300.

(18) The indicated horse-power of an engine is 10.6. The actual effective horse-power is 8.96. What per cent of the indicated is the actual horse-power? Ans. $84\frac{48}{53}\%$.

DENOMINATE NUMBERS.

15. A **Denominate Number** is a concrete number in which the unit of measure is established by law or custom. Such a number as 7 pounds, 6 feet, 8 inches.

16. When a denominate number is composed of units of but one denomination, as, for example 3 gallons, 5 feet, and 4 tons, it is called a **Simple Denominate Number.** If it contains units of two or more denominations that are related to each other, as, 2 feet 10 inches, 12 pounds 7 ounces, it is called a **Compound Denominate Number.**

17. **Reduction of Denominate Numbers** is the process of changing them from one denomination to another without changing the value. We may change from a higher to a lower or from a lower to a higher denomination. The former is accomplished by multiplication, the latter by division. For example:

4 feet = 48 inches.

48 inches = 4 feet.

In the first case we reduced feet to inches by multiplying by 12. In the second case we reduced inches to feet by dividing by 12.

18. A **Measure** is a standard unit, by which quantity, such as capacity, weight, and the like, is measured. For example, a pound is a standard unit.

MEASURES OF LENGTH.

19. We measure the length by applying to it a line of known length, such as a foot, and ascertaining how many times that line is contained. Such a line of known length is termed a *linear unit.* The following tables show the *measures of length,* and the relations between them.

LINEAR MEASURE.

12 inches (in.) = 1 foot (ft.)

3 feet = 1 yard (yd.)

$5\frac{1}{2}$ yards = 1 rod (rd.)
40 rods = 1 furlong (fur.)
8 furlongs = 1 mile (mi.)

Mi.	*Fur.*	*Rods.*	*Yds.*	*Ft.*	*In.*
1 =	8 =	320 =	1760 =	5280 =	63360
	1 =	40 =	220 =	660 =	7920
		1 =	$5\frac{1}{2}$ =	$16\frac{1}{2}$ =	198
			1 =	3 =	36
				1 =	12

19. To illustrate reduction take the following example:

Reduce 17 yards, 2 feet, 7 inches to inches.

17 = number of yards.
3

51 = number of feet in 17 yards.
2

53 = number of feet in 17 yards, 2 feet.
12

106
53

636 = number of inches in 17 yards, 2 feet.
7

643 = number of inches in 17 yards, 2 feet. 7 inches.

Therefore 643 inches is the answer.

We might obtain this more speedily by using the second table.

17 yards = 17 × 36 or 612 inches (since) 1 yd. = 3 ft. = 36 in.
2 feet = 2 × 12 or 24 inches.

Therefore in 17 yards, 2 feet, 7 inches are 612 + 24 + 7 or 643 inches.

21. Now suppose that we wished to change 643 inches to yards, feet and inches, *i. e.*, the reverse of the preceding process.

12)643
53 and 7 as a remainder.

Hence in 643 inches there are enough inches to make 53 feet and leave 7 inches remaining.

3)53
17 and 2 as a remainder.

Hence in 53 feet there are enough feet to make 17 yards, and 2 feet over.

Therefore in 643 inches there are 17 yards, 2 feet, and 7 inches.

It is usual to perform the process thus:

12 | 643 in.
3 | 53 ft. 7 in.
17 yds. 2 ft. 7 in. Ans.

22. The above has given us *measures of length*. If we wish to measure surface, we use *measures of surface*. These are as follows:

SQUARE MEASURE.

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd).
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A)
640 acres	= 1 square mile (sq. mi.)

We are now using a *known amount of surface* as a unit. Observing the table we see that instead of 12 square inches in a square foot, we have 12 × 12 or 144 square inches. Instead of 3 we have

3 × 3 or 9 square feet in a square yard, and so on. The following illustrates the reason for this:

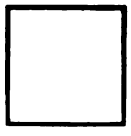


Fig. 1.

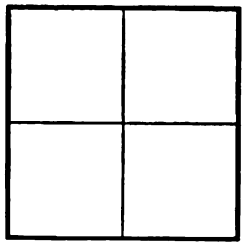


Fig. 2

Let the side of Fig. *1* be 1 unit, and that of Fig. *2*, two units in length. Now in Fig. *1* there is one unit of surface or one square unit. In Fig. *2*, in which each side is twice as long, however, there are 2 × 2 or 4 such units.

The following table gives further illustrations of square measure.

Sq. Mi.	*A.*	*Sq. Rd.*	*Sq. Yd.*	*Sq. Ft.*	*Sq. In.*
1 =	640 =	102,400 =	3,097,600 =	27,878,400 =	4,014,489,600
	1 =	160 =	4,840 =	43,560 =	6,272,640
		1 =	30¼ =	272¼ =	39,204
			1 =	9 =	1,296
				1 =	144

23. Reduction in square measure is a similar operation to that in linear and other measures.

Reduce 87 sq. yds. 4 sq. ft. to square feet.

87 (No. of sq. yds.).
9 (No. of sq. ft. in a sq. yd.).

783 (No. of sq. ft. in 87 sq. yds.).
4 (No. of sq. ft.).

787 (No. of sq. ft. in 87 sq. yds. 4 sq. ft.). Ans.

Reduce 787 sq. ft. to higher denominations.

9)787

87 sq. yds. 4 sq. ft.

NOTE. Do not neglect in using this measure to prefix sq. to the yds., ft., in., etc.

24. Measures of *volume* are called cubic measures.

CUBIC MEASURE.

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)

27 cubic feet = 1 cubic yard (cu. yd.)

128 cubic feet = 1 cord (cd.)

NOTE. 16 cu. ft. = 1 cord ft. (cd. ft.) A pile of wood 8 feet long, 4 feet wide, and 4 feet high contains a cord. The cord foot is 1 foot in length of such a pile.

NOTE. A perch of stone or masonry is $16\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and 1 foot high. In some localities a perch of stone is considered as 22 cubic feet. In others 18 cubic feet. We will consider it to be $24\frac{3}{4}$ cubic feet.

25. Observing this table we see that 1728 is $12 \times 12 \times 12$ and 27 is $3 \times 3 \times 3$. The following shows the reason for this:

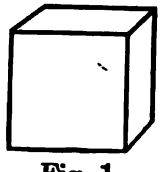


Fig. 1.

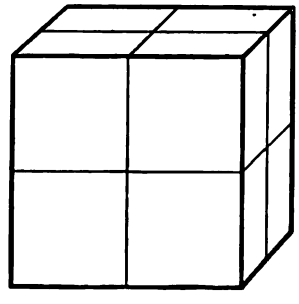


Fig. 2.

In Fig. *1.* we have a block which we will consider as unit volume, each of its edges being a unit of length. Each edge of Fig. *2* is two units in length, and in its lower layer are four such

units of volume, and in the upper layer are also four such units, making a total of eight units.

The volume of Fig. *1* is 1 × 1 × 1 or 1 unit.

The volume of Fig. *2* is 2 × 2 × 2 or 8 units.

26. We now come to *Measures of Weight.*

AVOIRDUPOIS WEIGHT.

16 ounces (oz.) = 1 pound (lb.)
100 pounds = 1 hundred-weight (cwt.)
20 hundred-weight or 2,000 pounds = 1 ton (T.)

T.	*Cwt.*	*Lb.*	*Oz.*
1 =	20 =	2000 =	32,000
	1 =	100 =	1,600
		1 =	16

NOTE. There are 16 drams in an Avoirdupois ounce. 25 pounds are sometimes called a *quarter.*

LONG TON TABLE.

16 ounces = 1 pound.
112 pounds = 1 hundred weight.
20 cwt. or 2,240 lbs. = 1 ton.

The long ton table is not to be used unless especially designated.

NOTE. The quarter of the long hundred-weight is 28 pounds.

27. Change 2 tons, 3 hundred-weight, 49 pounds to pounds.

2 tons = 4000 lbs.
3 cwt. = 300 lbs.
49 lbs. = 49 lbs.
2 T. 3 cwt. 49 lbs. = 4,349 lbs.

By the reverse operation 4,349 lbs. = 2 T. 3 cwt. 49 lbs.

28. It is well to observe the following tables, although they are not absolutely essential.

TROY WEIGHT.

24 grains (gr.) = 1 pennyweight (pwt.)
20 pennyweights = 1 ounce (oz.)
12 ounces = 1 pound (lb.)
also $3\frac{1}{8}$ grains = 1 carat (K.)

APOTHECARIES' WEIGHT.

20 grains (gr.) = 1 scruple (℈)
3 scruples = 1 dram (ʒ)
8 drams = 1 ounce (℥)
12 ounces = 1 pound (℔)

SURVEYORS' MEASURE.

LINEAR.

7.92 inches (in.) = 1 link (l.)
25 links = 1 rod (rd.)
100 links, 4 rods, 66 feet each = 1 chain (ch.)
80 chains = 1 mile (mi.)

SQUARE MEASURE.

10 square chains = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)
1 square mile = 1 section (sec.)
36 sections = 1 township (Tp.)

29. We now come to *Measures of Capacity*.

LIQUID MEASURE.

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
$31\frac{1}{2}$ gallons = 1 barrel (bbl.)
2 barrels or 63 gallons = 1 hogshead (hhd.)

Hhd.	*Bbl.*	*Gal.*	*Qt.*	*Pt.*	*Gi.*
1 =	2 =	63 =	252 =	504 =	2,016
	1 =	$31\frac{1}{2}$ =	126 =	252 =	1,008
		1 =	4 =	8 =	32
			1 =	2 =	8
				1 =	4

Change 1 hhd. 1 bbl. 7 gal. 2 qts. to quarts.

1 hhd. = 63 × 4 or 252 qt.
1 bbl. = $31\frac{1}{2}$ × 4 or 126 qt.
7 gal. = 7 × 4 or 28 qt.
2 qt. = 2 qt.

Hence the result is 408 qts.

30. Another measure of capacity is dry measure.

DRY MEASURE.

2 pints (pts.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

Bu.	*Pk.*	*Qt.*	*Pt.*
1 =	4 =	32 =	64
	1 =	8 =	16
		1 =	2

Change 7 bu. 3 pk. to quarts.

7 bu. = 224 qts.
3 pk. = 24 qts.
7 bu. 3 pk. = 248 qts.

31. We now come to *Measure of Time.*

MEASURE OF TIME.

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (d.)
7 days = 1 week (w.)
365 days / 12 months (mo.) = 1 year (yr.)
366 days = 1 leap year.
100 years = 1 century.

It is customary to consider a month as 30 days

How many seconds in a week?

24 (No. of hrs. in a day).
7 (No. of days in a week).

168 (No. of hrs. in a week).
60 (No. of min. in an hour).

10,080 (No. of min. in a week).
60 (No. of sec. in a min.).

604,800 (No. of sec. in a week). **Ans.**

32. Measure of Angles or Arcs.

60 seconds (″) = 1 minute (′).
60 minutes = 1 degree (°).
90 degrees = 1 right angle or quadrant (L).
360 degrees = 1 circumference.

MEASURE OF MONEY.

10 mills (m.) = 1 cent (ct.).
10 cents = 1 dime (d.).
10 dimes = 1 dollar ($).
10 dollars = 1 eagle (E.).

NOTE. The following are important figures:

1 U. S. standard gallon contains 231 cubic inches and weighs approximately 8.33 pounds.

1 cubic foot of water contains 7.48 U. S. standard gallons (approximately) and weighs $62\frac{1}{2}$ pounds.

1 bushel contains 2,150.4 cubic inches.

EXAMPLES FOR PRACTICE.

Reduce

(1) 27 yd. 2 ft. 11 in. to inches — **Ans.** 1,007 in.
(2) 18 gal. 3 qt. 1 pt. 3 gi. to gills — **Ans.** 607 gi.
(3) 18 hr. 20 min. 35 sec. to seconds — **Ans.** 66,035 sec.
(4) 4 T. 7 cwt. 35 lb. 9 oz. to ounces — **Ans.** 139,769 oz.

(5) 8 bu. 3 pk. 7 qt. 1 pt. to pints — Ans. 575 pt.
(6) 5 sq. yd. 4 sq. ft. 100 sq. in. to sq. in. — Ans. 7,156 sq. in.
(7) 8 A. 25 sq. rd. 4 sq. yd. to sq. yd. — Ans. 39,480¼ sq. yd.
(8) 4 cu. yd. 6 cu. ft. 714 cu. in to cu. in. — Ans. 197,706 cu. in.
(9) 7 mi. 100 rd. 13 ft. 6 in. to inches — Ans. 463,482 in.

Reduce to higher denominations

(10) 2,456 inches — Ans. 12 rd. 2 yd. 8 in.
(11) 5,937 square inches — Ans. 4 sq. yd. 5 sq. ft. 33 sq. in.
(12) 216,840 cubic inches — Ans. 4 cu. yd. 17 cu. ft. 840 cu. in.
(13) 5,983 pounds — Ans. 2 T. 19 cwt. 83 lb.
(14) 4,213 gills — Ans. 2 hhd. 5 gal. 2 qt. 1pt. 1 gi.
(15) 4,683 pints — Ans. 73 bu. 5 qt. 1 pt.

33. RULE. To change a compound number to a simple number of a lower denomination: Multiply the number of the highest denomination by the number of units it takes of the next lower denomination to make one of that higher, and to the product add the given number of the next lower denomination. Multiply this sum in the same way, and proceed in this manner until the given number is changed to units of the required denomination.

RULE. To change a simple number to a compound number of higher denominations: Divide the given number by the number of units it takes of its denomination to make one of the next higher. Set aside the remainder and divide in the same manner the quotient thus obtained, and proceed in this way until the required denomination is reached. The last quotient of the several remainders is the result sought.

OPERATIONS WITH DENOMINATE NUMBERS.

34. There are two methods by which to add, subtract, multiply or divide denominate numbers. One is to reduce the given numbers to the lowest denominations given in the example, and then perform the required process, and reduce the result to higher denominations. The other is to perform the process on the numbers as they stand, making the necessary reductions during the operation. The following furnish examples of the two methods.

35. Addition. Find the sum of 3 m. 182 rd. 4 yd. 2 ft., 304 rd. 1 ft., and 5 m. 76 rd. 4 yd. 2 ft.

3 m. 182 rd. 4 yd. 2 ft. = 15,840 + 3,003 + 12 + 2 = 18,857 ft.
304 rd. 0 yd. 1 ft. = 5,016 + 1 = 5,017 ft.
5 m. 76 rd. 4 yd. 2 ft. = 26,400 + 1,254 + 12 + 2 = 27,668 ft.

51,542 ft.

51,542 ft. = 9 m. 243 rd. 4 yd. 0 ft. 6 in.

NOTE. We have $\frac{1}{2}$ ft. or 6 in. left because the odd number of rods (each $16\frac{1}{2}$ ft.) leaves $\frac{1}{2}$ ft. *i.e.* we divide an integral number of feet by the mixed number $16\frac{1}{2}$. As the quotient is an odd number, 243, we have a mixed number, $12\frac{1}{2}$, as a remainder, $12\frac{1}{2}$ ft. equal 4 yd. 6 in.

The other method is the following:

m.	*rd.*	*yd.*	*ft.*
3	182	4	2
	304	0	1
5	76	4	2
9	243	$3\frac{1}{2}$	2

The sum of the right hand column (ft.) is 5 ft. which equals 1 yd. 2 ft. We write down the 2 and add the 1 into the next column.

The sum of the next column (yd.) is, with the 1 carried over into it, 9 yds. Since $5\frac{1}{2}$ yd. make a rod we put down the difference between 9 and $5\frac{1}{2}$ which is $3\frac{1}{2}$, and add 1 into the rd. column.

The sum of the rd. column with this 1 added is 563 rd. Of this 563 rd. we take 320 rd. to add a mile into the m. column, which thus becomes 9 m. Our result is, therefore, 9 m. 243 rd.

$3\frac{1}{2}$ yd. 2 ft. We do not want the fraction of a yard in our answer, so we reduce it to feet. As a yard is 3 ft., $\frac{1}{2}$ yd. equals $1\frac{1}{2}$ ft. This gives $3\frac{1}{2}$ in the ft. column. But 3 ft. equal 1 yd., so we write 4 in the yd. column, and, instead of writing $\frac{1}{2}$ in the ft. column, we write 6 in. in a new column.

Take the following example: Add 35 bu. 3 pk. 7 qt., 9 bu. 2 pk. 3 qt., 48 bu. 3 pk. 7 qt., and 2 bu. 2 qt.

bu.	*pk.*	*qt.*
35	3	7
9	2	3
48	3	7
2	0	2
96	2	3

The sum of the qt. column is 19 qt. or 2 pk. 3 qt. Adding in the 2 pk. the sum of the pk. column is 10 pk. or 2 bu. 2 pk. Adding in the 2 bu. we have 96 in the bu. column. Thus the result is 96 bu. 2 pk. 3 qt.

36. Subtraction. The same methods are here used except that in the second method we take away from the higher denominations to increase the lower ones.

Subtract 3 pk. 1 qt. 1 pt. from 1 bu. 2 pk. 4 qt.

By first method:

1 bu. 2 pk. 4 qt. $= 64 + 32 + 8 =$ 104 pt.
3 pk. 1 qt. 1 pt. $= 48 + 2 + 1 =$ 51 pt.
53 pt.

53 pt. = 26 qt. 1 pt. = 3 pk. 2 qt. 1 pt. Ans.

By second method:

0	6	3	2	minuend reduced to allow subtraction.
bu.	*pk.*	*qt.*	*pt.*	
1	2	4	0	minuend
	3	1	1	subtrahend
	3	2	1	

Hence the difference is 3 pk. 2 qt. 1 pt.

NOTE. The upper row shows how we took 1 qt. to make 2 pt. leaving 3 qt., and how 1 bu. was taken in order to add 4 pk. to 2 pk. so as to make 6 and allow the subtraction of 3 pk. This line of figures was inserted only to show the method, and it should ordinarily be left out.

37. Multiplication. Multiply 14 gal. 3 qt. 1 pt. by 7.

By first method:

14 gal. 3 qt. 1 pt. $= 112 + 6 + 1 = 119$ pt.
119 pt. $\times$ 7 $= 833$ pt. $= 416$ qt. 1 pt. $= 104$ gal. 1 pt. Ans.

By second method:

14 gal. 3 qt. 1 pt. multiplicand.
7 multiplier.

104 gal. 0 qt. 1 pt. product.

By multiplying 14, 3, and 1 by 7 we obtain the following:

unreduced product	98 gal. 21 qt. 7 pt.
with pints reduced	98 gal. 24 qt. 1 pt.
entirely reduced (product as before)	104 gal. 0 qt. 1 pt.

These figures show the process of reduction which is carried on when obtaining the answer.

38. Division. Divide 14 gal. 3 qt. 1 pt. by 4.

By first method:

14 gal. 3 qt. 1 pt. $= 119$ pt.

4)119 pt.
$29\frac{3}{4}$ pt. $=$ 14 qt. $1\frac{3}{4}$ pt. $=$ 3 gal. 2 qt. $1\frac{3}{4}$ pt.

By second method:

gal.	*qt.*	*pt.*
4)14	3	1
3	2	$1\frac{3}{4}$

Hence the quotient is 3 gal. 2 qt. $1\frac{3}{4}$ pt.

NOTE. After dividing 14 gal. by 4, the remainder is 2 gal. This remainder is added to 3 qt. making $8 + 3$ or 11 qt. Dividing 11 by 4 the remainder is 3. As there are 2 pt. in 1 qt., we have $3 \times 2 + 1$ or $6 + 1$ or 7 pt. Dividing by 4 we obtain $1\frac{3}{4}$ pt.

NOTE. The second method is generally the easier.

39. RULE. Reduce the given numbers to the lowest denomination stated in the example. Perform the required operation, (addition, subtraction, multiplication or division,) and then if required reduce the result to a higher denomination.

Or: Perform the required operation on the numbers as they stand, reducing the number to other denominations whenever necessary.

EXAMPLES FOR PRACTICE.

Find the sum of:

(1) 10 yd. 2 ft. 10 in.; 15 yd. 1 ft. 9 in.; 8 yd. 2 ft. 7 in.; 18 yd. 1 ft. 11 in.; 16 yd. 2 ft. 8 in. Ans. 70 yd. 2 ft. 9 in.

(2) 12 A. 35 sq. rd.; 14 A. 110 sq. rd.; 15 A. 132 sq. rd.; 11 A. 96 sq. rd.; 25 A. 100 sq. rd. Ans. 79 A. 153 sq. rd.

(3) 5 T. 6 cwt. 14 lb. 10 oz.; 7 T. 15 cwt. 36 lb. 15 oz.; 17 T. 5 cwt. 84 lb. 12 oz.; 70 T. 9 cwt. 94 lb. 11 oz.
Ans. 100 T. 17 cwt. 31 lb.

From

(4) 12 gal. 2 qt. 1 pt. 2 gi. take 6 gal. 3 qt. 1 pt. 3 gi.
Ans. 5 gal. 2 qt. 1 pt. 3 gi.

(5) 15 yd. 2 ft. 7 in. take 4 yd. 2 ft. 10 in.
Ans. 10 yd. 2 ft. 9 in.

(6) 25 T. 8 cwt. 75 lb. 10 oz. take 10 T. 11 cwt. 35 lb. 15 oz. Ans. 14 T. 17 cwt. 39 lb. 11 oz.

Multiply

(7) 3 hr. 20 min. 35 sec. by 5. Ans. 16 hr. 42 min. 55 sec.

(8) 2 T. 5 cwt. 48 lb. 15 oz. by 8.
Ans. 18 T. 3 cwt. 91 lb. 8 oz.

(9) 12 cu. yd. 15 cu. ft. 1115 cu. in. by 6.
Ans. 75 cu. yd. 12 cu. ft. 1506 cu. in.

Divide

(10) 15 bu. 3 pk. 5 qt. 1 pt. by 4.
Ans. 3 bu. 3 pk. 7 qt. $\frac{3}{4}$ pt.

(11) 23 cwt. 68 lb. 10 oz. by 5. Ans. 4 cwt. 73 lb. $11\frac{3}{5}$ oz.

(12) 15 rd. 4 yd. 2 ft. 8 in. by 5. Ans. 3 rd. 2 ft. $11\frac{1}{5}$ in.

(13) The weights of coal fed to a furnace on four successive days are as follows: 1 T. 7 cwt. 80 lb.; 1 T. 6 cwt. 96 lb.; 1 T. 5 cwt. 87 lb.; 1 T. 9 cwt. 18 lb. What is the total amount used during the four days? Ans. 5 T. 9 cwt. 81 lb.

(14) A bin contains 8 T. 7 cwt. If 1 T. 17 cwt. 88 lb. are taken from it at one time and 2 T. 3 cwt. 19 lb. at another time, what weight remains? Ans. 4 T. 5 cwt. 93 lb.

(15) A tank contains 2,747 gal. 3 qt. What will be the amount of water in a tank 4 times as large? Ans. 10,991 gal.

(16) How many iron rails, each 16 feet long, will be required to lay a railroad track 26 m. long? Ans. 17,160 rails.

(17) The aggregate weight of 41 hhd. of sugar is 19 T. 6 cwt. 22 lb. What is the average weight? Ans. 9 cwt. 42 lb.

INVOLUTION.

40. **Involution** is the process of multiplying any number by itself several times. The product thus obtained is called a power of the number.

41. In order to show how many times the number is to be used as a factor, we write a small figure to the right and a little above the number, as 3^4. This small number is called an **Exponent.** It shows *to what power* the number is to be raised. Thus 3^4 means the fourth power of 3 or $3 \times 3 \times 3 \times 3$ or 81, *i.e.*, 3 is taken 4 times as a factor.

The second power of a number is called its square. Thus 4 is the square of 2, for $2 \times 2 = 4$.

The third power of a number is called its cube. Thus 8 is the cube of 2, for $2 \times 2 \times 2 = 8$.

42. The following furnishes examples of powers:

$3^2 = 3 \times 3 = 9$ (square or second power of 3).

$3^3 = 3 \times 3 \times 3 = 27$ (cube or third power of 3).

$3^4 = 3 \times 3 \times 3 \times 3 = 81$ (fourth power of 3).

$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$ (fifth power of 3),

and so on.

43. Examples in finding various powers.

What is the square or second power of 15?

$$15 \times 15 = 225 \text{ Ans.}$$

What is the cube or third power of 21?

$$21 \times 21 \times 21$$

$$\begin{array}{r} 21 \\ 21 \\ \hline 21 \\ 42 \\ \hline 441 \\ 21 \\ \hline 441 \\ 882 \\ \hline 9261 \text{ Ans.} \end{array}$$

What is the fifth power of 1.3?

$$1.3 \times 1.3 \times 1.3 \times 1.3 \times 1.3$$

1.3
1.3
39
13
1.69
1.3
507
169
2.197
1.3
6591
2197
2.8561
1.3
85683
28561
3.71293 Ans.

NOTE. As 1.3 contains one decimal place, and is used as a factor five times, there must be five decimal places in the answer.

What is the cube of .71?

$$.71 \times .71 \times .71$$

.71
.71
71
497
.5041
.71
5041
35287
.357911 Ans.

What is the fourth power of $\frac{3}{5}$?

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{3 \times 3 \times 3 \times 3}{5 \times 5 \times 5 \times 5} = \frac{81}{625} \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

Raise to the powers indicated.

(1) 47^2 — Ans. 2,209

(2) 781^3 — Ans. 476,379,541

(3) 71^2 — Ans. 5,041

(4) 2.61^2 — Ans. 6.8121

(5) $.131^3$ — Ans. .002248091

(6) $.0303^2$ — Ans. .00091809

(7) $(\frac{4}{7})^5$ — Ans. $\frac{1024}{16807}$

(8) $(3\frac{2}{3})^5$ — Ans. $662\frac{185}{243}$

EVOLUTION.

44. Evolution is the reverse of involution. It is the process of finding the number when *some power of the number* is given.

45. One of the equal factors which make up a number is called a *root* of that number. If a certain number is used twice as a factor in order to produce another number, then the number first mentioned is the **Square Root** of the second. If the first number must be used three times as a factor, it is the **Cube Root** of the second number. If used five times it is the fifth root. Thus 2 is the square root of 4 (for $2 \times 2 = 4$), the cube root of 8 (for $2 \times 2 \times 2 = 8$), and the fifth root of 32 (for $2 \times 2 \times 2 \times 2 \times 2 = 32$.

46. The **Radical Sign** $\sqrt{\ }$ placed before a number, shows that some root of it is to be taken, *which* root is shown by the **Index** which is a small figure placed above and to the left of the radical sign. When no index is written, the sign indicates that the square root is to be taken. The following examples show the meaning of sign and index.

$$\sqrt{81} = 9 \text{ for } 9 \times 9 = 81$$

$$\sqrt[4]{81} = 3 \text{ for } 3 \times 3 \times 3 \times 3 = 81$$

$$\sqrt[3]{27} = 3 \text{ for } 3 \times 3 \times 3 = 27$$

$$\sqrt[5]{32} = 2 \text{ for } 2 \times 2 \times 2 \times 2 \times 2 = 32$$

SQUARE ROOT.

47. From the study of evolution we learn that if a number is used twice as a factor to produce another number it is called the square root of the number. Thus, to obtain 25 we use 5 twice as a factor, thus $5 \times 5 = 25$. We call 5 the square root of 25. Similarly $8 \times 8 = 64$, or 8 is the square root of 64.

A	B	A	B	A	B
1^2 =	1	10^2 =	100	100^2 =	10,000
3^2 =	9	12^2 =	144	317^2 =	100,489
5^2 =	25	25^2 =	625	500^2 =	250,000
9^2 =	81	99^2 =	9,801	759^2 =	576,081

A	B
$1{,}000^2$ =	1,000,000
$1{,}144^2$ =	1,308,736
$1{,}419^2$ =	2,013,561

From the above table we see that when the numbers in the columns marked B have either one or two figures, those opposite them in the A columns have one figure. When there are three or four places in the columns marked B we have two places in the corresponding A columns. When there are five or six places in the B columns, there are three in the corresponding A columns.

From this we see that, if a numerical expression be separated into periods of two places each, beginning with the units' figure, the number of periods (the left hand period may contain but one figure) will be the same as the number of figures in the square root.

Thus, to find how many figures there are in the square root of 900 we write it $\overset{\frown}{9}\ \overset{\frown}{00}$, and as there are two periods there will be 2 figures in the square root. In like manner there will be 4 figures in the square root of $\overset{\frown}{31}\ \overset{\frown}{50}\ \overset{\frown}{57}\ \overset{\frown}{69}$ since it can be divided into four periods of two figures each.

Take for example 36, it has one period and one figure in its square root. $\overset{\frown}{11}\ \overset{\frown}{08}\ \overset{\frown}{89}$ has three periods and three figures in its square root.

48. Perhaps the best method of explaining the process of finding the square root of any number is by an example.

Find the square root of 256.

We first divide it into periods of two figures each commencing at the decimal point, to find the number of figures in the root, as $\widehat{2}\ \widehat{56}$.

```
                2 56( 16
                1
              ------
20              1 56
 6
---
26              1 56
              ------
                  0
```

First, find the greatest number whose square is equal to or less than 2. We may have to do this by trial until we learn the squares of the numbers 1 to 9 inclusive. The number is evidently 1 since $1^2 = 1$ and the square of the next number 2, is 4. Then we place the square of 1 under the 2 and subtract, bringing down the next period 56, making 156 as our new dividend. We then double the part of the root already found and after annexing a cipher we place the number at the left and use it as our trial divisor. We find that 20 is our trial divisor, since $1 \times 2 = 2$ and after annexing a cipher it becomes 20. 20 is contained in 156, 6 times; it is also contained 7 times, but when we add 7 to our trial divisor and multiply it by 7 the result (189) is greater than 156. Therefore 7 cannot be used.

We add 6 to the 20 making our trial divisor 26. We then multiply this sum by 6 and place the product (156) under the 156 and subtract. We place the six in the root making our answer 16.

Another example. Find the square root of 185761.

```
                18 57 61( 431
                16
              ------
 80              2 57
  3
---
 83             2 49
              ------
860                8 61
  1
---
861                8 61
                 ------
                     0
```

First, point off the number into periods of two figures each commencing at the decimal point. Then we find the largest number whose square is equal to or less than 18; we find it to be 4 so we square 4 and place 16 under the 18.

After subtracting and bringing down the next period, as in the previous example, we find that 257 is our next dividend. We then double the part of the root already found (4), and annex a cipher, which makes our first trial divisor 80. By trial we find 80 is contained 3 times in 257 so we add 3 to the 80, multiply this sum by 3 and place the product (249) under the 257 and subtract; we place the 3 as the second figure of the root. Bringing down the last period we have 861 for our new dividend.

We then double the part of the root already found (43), and annex a cipher to it making 860 as our trial divisor. 860 is contained in 861 once, so we add 1 to our trial divisor making our complete divisor 861 and place the 1 as the third figure of the root. We now multiply 861 by 1 and place it under the 861 and subtract. As there is no remainder we do not continue the process. The answer is the root, 431.

In case the number is a decimal, or a whole number and a decimal, point off into periods of two figures each commencing at the decimal point. For example, if we wish to find the square root of 18673.8910 we mark off the periods, thus, $\overset{\frown}{1}\ \overset{\frown}{86}\ \overset{\frown}{73}.\overset{\frown}{89}\ \overset{\frown}{10}$ and proceed as before, extracting the root of the decimal the same as with whole numbers.

49. From the above we may formulate rules to follow when finding the square root of numbers.

Separate the given number into periods of two figures each, beginning at the decimal point. If there are whole numbers and decimals mark off the periods to the left for the whole numbers, and mark off the periods to the right for the decimal, annexing ciphers to the decimal if necessary.

Find the greatest number whose square is contained in the left hand period. This figure will be the first figure in the root. Subtract the square of this number from the left hand period and bring down the next period, to form, with the remainder, the new dividend.

Double the part of the root already found and after annexing a cipher place it on the left for a trial divisor. Ascertain how many times this divisor is contained in the dividend and write it as the next figure in the root. Then add the number just placed in the root to the trial divisor and multiply the new divisor by the last figure in the root. Subtract this product from the dividend.

To the remainder annex the next period for the new dividend. Double the part of the root already found, annex a cipher and use the result as the trial divisor. Continue the operation until all the periods are used.

50. These rules are applied in the following examples.

Find the square root of 2,556,801.

$$\begin{array}{r} \widehat{2}\,\widehat{55}\,\widehat{68}\,\widehat{01}\,(1599 \\ 1 \\ \hline 25)\,155 \\ 125 \\ \hline 309)\,3068 \\ 2781 \\ \hline 3189)\,28701 \\ 28701 \\ \hline \end{array}$$

Find the square root of 406457.2516.

$$\begin{array}{r} \widehat{40}\,\widehat{64}\,\widehat{57}.\widehat{25}\,\widehat{16}\,(637.54 \\ 36 \\ \hline 123)\,464 \\ 369 \\ \hline 1267)\,9557 \\ 8869 \\ \hline 12745)\,68825 \\ 63725 \\ \hline 127504)\,510016 \\ 510016 \\ \hline \end{array}$$

Find the square root of .001225

$$
\begin{array}{r}
\widehat{.00}\,\widehat{12}\,\widehat{25}\,(.035 \\
9 \\
\hline
65)\ 325 \\
325 \\
\hline
\end{array}
$$

Find the square root of 3, to 5 decimal places.

$$
\begin{array}{r}
\widehat{3}.\widehat{00}\,\widehat{00}\,\widehat{00}\,\widehat{00}\,\widehat{00}\,(1.73205+ \\
1 \\
\hline
27)200 \\
189 \\
\hline
343)\ 1100 \\
1029 \\
\hline
3462)\ 7100 \\
6924 \\
\hline
346405)\ 1760000 \\
1732025 \\
\hline
27975
\end{array}
$$

EXAMPLES FOR PRACTICE.

What is the square root of

(1) 516,961 — Ans. 719

(2) 23804641 — Ans. 4879

(3) 538 to three decimal places. — Ans. 23.194+

(4) .3364 — Ans. .58

(5) .120409 — Ans. .347

(6) 1159.4025 — Ans. 34.05

(7) 2 to four decimal places. — Ans. 1.4142+

(8) 769 to four decimal places. — Ans. 27.7308+

(9) 998001 — Ans. 999.

(10) A square piece of land contains 734449 sq. ft. What is the length of each of its sides? — Ans. 857 ft.

(11) A square field contains 1,016,064 sq. ft. What is the length of each side? — Ans. 1008 ft.

CUBE ROOT.

51. Cube root is the third root of any power. To find the cube root of a number, is to resolve it into 3 equal factors.

First mark off into periods of three figures each, beginning (as we did with square root), at the decimal point; this determines the number of places in the root.

For example, there will be two figures in the cube root of 74088 because there are two periods thus $\widehat{74}\ \widehat{088}$.

Find the cube root of 50,653.

$\widehat{50}\ \widehat{653}$ (37

27

23 653

$3^2 \times 300 = 2700$

$3 \times 7 \times 30 = 630$

$7^2 = 49$

3379 23 653

0

After separating the number into periods of three figures each we find the greatest number whose cube is less than 50. We do this by trial as we did in square root. We find it to be 3 and its cube is 27. We place 3 as the first figure of the root and its cube (27) under the first period. Subtracting 27 from 50 and bringing down the next period gives us 23653 for our new dividend.

To find our trial divisor we square the part of the root already found and multiply it by 300. Thus $3^2 \times 300 = 2700$. This is our trial divisor. We next ascertain how many times 2700 is contained in 23653. It is contained 7, 8 or 9 times but as either 8 or 9 would produce too great a number we use the 7. We place 7 as the second figure of the root. Next multiply the part of the root already found by 7 and by 30 and add this ($3 \times 7 \times 30 = 630$) to the trial divisor. We square the last figure in the root and add it to the trial divisor making ($2700 + 630 + 49$) 3379 our complete divisor. Multiplying our new divisor 3379 by 7 we find it equal to our dividend 23653. The answer is the root 37.

52. Another example. Find the cube root of 461889917.

		$\overset{\frown}{461}\ \overset{\frown}{889}\ \overset{\frown}{917}$ (773
		343
$7^2 \times 300$	= 14700	118 889
$7 \times 7 \times 30$	= 1470	
7^2	= 49	
	16219	113 533
$77^2 \times 300$	= 1778700	5 356 917
$77 \times 3 \times 30$	= 6930	
3^2	= 9	
	1785639	5 356 917
		0

We first separate the number into periods of three figures each, commencing at the decimal point. Find the greatest number whose cube is equal to or less than the first period. This is accomplished by trial as in square root. In the accompanying example we find this to be 7, and we place the 7 at the right as the first figure of the root, and its cube ($7 \times 7 \times 7$) 343, under the first period 461. Subtract and bring down the next period 889, which gives us 118889 as our new dividend.

Then we square the part of the root already found and multiply it by 300; thus, $7^2 \times 300 = 14700$, and this is our trial divisor. Next we ascertain how many times it will be contained in 118889. We find it to be 7 so we place 7 as the second figure in the root. Now multiply the first figure by the last figure of the root found, and by 30; thus, $7 \times 7 \times 30 = 1470$; place this product under the trial divisor. Square the last figure found, ($7 \times 7 = 49$) and place this under the other two products; add the three results and multiply the sum by the figure just found, and place the product under the dividend. Subtract and bring down the last period and we have 5356917 as our last dividend.

Square the part of the root already found (77) and multiply by 300; thus, $77^2 \times 300 = 1778700$. Taking this as our new trial divisor we ascertain how many times it is contained in 5356917. As we find it to be 3, we multiply the part of the root already found (77) by 3 and by 30, and place the product which

is 6930 under the new trial divisor. Place the 3 as the third figure of the root. Square it and place this product also under the trial divisor. Add the three products and multiply the result by the figure just found; place this product under the dividend and subtract. We obtain as our answer 773.

We may formulate rules for finding cube root.

53. Separate the given number into periods of three figures each. The period to the left often has but one or two figures. If the number contains a decimal, point off the periods to the right of the decimal point annexing ciphers if necessary.

Find the greatest number whose cube is not greater than the left hand period. Place this number as the first figure of the root and its cube under the first period. Subtract and bring down the second period for the new dividend.

Square the part of the root already found and multiply it by 300. Ascertain how many times this trial divisor is contained in the dividend and write the result as the next figure of the root. Multiply the last figure of the root by the preceding part and multiply the product by 30. Square the last figure of the root and add both products to the trial divisor. The sum is the new divisor.

Multiply the new divisor by the last figure of the root and subtract the product from the dividend. Bring down the next period to form, with the remainder, another dividend.

Find trial and complete divisors as before and continue the process until all the periods have been used.

Find the cube root of 126732.947167.

$$
\begin{array}{rrrrrl}
 & & & & \overset{\frown}{1}\overset{\frown}{2}\overset{\frown}{6}\overset{\frown}{7}\overset{\frown}{3}\overset{\frown}{2}.\overset{\frown}{9}\overset{\frown}{4}\overset{\frown}{7}\overset{\frown}{1}\overset{\frown}{6}\overset{\frown}{7} & (50.28 \\
 & & & & 125 & \\
 & 50^2 & \times 300 = & 750000 & 1732947 & \\
50 \times & 2 & \times 30 = & 3000 & & \\
 & 2^2 & = & 4 & 1506008 & \\
 & & & 753004 & & \\
 & 502^2 & \times 300 = & 75601200 & 226939167 & \\
502 \times & 3 & \times 30 = & 45180 & & \\
 & 3^2 & = & 9 & 226939167 & \\
 & & & 75646389 & 0 &
\end{array}
$$

NOTE. On bringing down the second period, 732, we readily see that 1732 is too small to contain $5 \times 5 \times 300$, so we write a 0 in the root, and proceed to determine the divisor for 1732947. The decimal point merely decides where the spacing into periods of three figures shall begin.

Find the cube root of 5 to five decimal places.

$$\begin{array}{rrrl}
 & & & \overset{\frown}{5}.\overset{\frown}{000}\overset{\frown}{000}\overset{\frown}{000}\overset{\frown}{000}\overset{\frown}{000}(1.70997+ \\
 & & & 1 \\
 & 1^2 \times 300 & = 300 & 4000 \\
1 \times 7 \times & 30 & = 210 & \\
 & 7^2 & = 49 & 3913 \\
 & & 559 & \\
 170^2 \times & 300 & = 8670000 & 87000000 \\
170 \times 9 \times & 30 & = 45900 & \\
 & 9^2 & = 81 & 78443829 \\
 & & 8715981 & \\
 1709^2 \times & 300 & = 876204300 & 8556171000 \\
1709 \times 9 \times & 30 & = 461430 & \\
 & 9^2 & = 81 & 7889992299 \\
 & & 876665811 & \\
 17099^2 \times & 300 & = 87712740300 & 666178701000 \\
17099 \times 7 \times & 30 & = 3590790 & \\
 & 7^2 & = 49 & 614014317973 \\
 & & 87716331139 & \\
 & & & 52164383027
\end{array}$$

EXAMPLES FOR PRACTICE.

Find the cube root of

(1) 185193 — Ans. 57

(2) 660776311 — Ans. 871

(3) 8.144865728 — Ans. 2.012

(4) .075686967 — Ans. .423

Find cube root to four decimal places of

(5) 2. — Ans. 1.2599+

(6) What should be the length of each side of a cubical cistern in order that it may hold 2744 cubic feet of water?

Ans. 14.

(7) What is the length of the edge of a cubical box that contains 46656 cubic inches, considering the thickness of each side to be 1 inch. — Ans. 3 ft. 2 ins.

ROOTS OF FRACTIONS.

54. When we wish to obtain some root of a common fraction, we find the required root of the numerator and write it over the same root of the denominator. We would not, however, pursue this method unless numerator and denominator were perfect powers. If they were not, the most correct and usually the easiest method, is to reduce the common fraction to a decimal fraction, and then find the required root.

55. Let us take some illustrations:

Find the square root of $\frac{16}{49}$.

$$\sqrt{\frac{16}{49}} = \frac{\sqrt{16}}{\sqrt{49}} = \frac{4}{7}$$

Find the square root of $\frac{169}{250}$ to three decimal places.

$$\begin{array}{r|l} 250\overline{)169000} & .\overset{\frown}{67}\overset{\frown}{60}\overset{\frown}{00}(.822+ \\ .676 & 64 \\ & \hline 162)360 \\ & 324 \\ & \hline 1642)3600 \\ & 3284 \end{array}$$

$$\text{Hence } \sqrt{\frac{169}{250}} = \sqrt{.676} = .822+$$

56. Find the cube root of $\frac{8}{27}$.

$$\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$$

Find the cube root of $\frac{3}{8}$ to two decimal places.

$$\begin{array}{r|l} 8\overline{)3.000} & .\overset{\frown}{37}\overset{\frown}{50}\overset{\frown}{00}\overline{).72} \\ .375 & 343 \\ 7\times 7\times 300 = 14700 & 32000 \\ 7\times 2\times 30 = 420 & \\ 2\times 2 = 4 & 30248 \\ 15124 & \end{array}$$

$$\text{Hence } \sqrt[3]{\frac{3}{8}} = \frac{\sqrt[3]{3}}{\sqrt[3]{8}} = \sqrt[3]{.375} = .72$$

NOTE. To find a root of a mixed number change it to a decimal.

EXAMPLES FOR PRACTICE.

Find the square root of

(1) $\frac{49}{169}$ Ans. $\frac{7}{13}$

(2) $\frac{27}{15625}$ Ans. .041+

Find the cube root of

(3) $\frac{64}{125}$ Ans. $\frac{4}{5}$

(4) $\frac{729}{1000}$ Ans. .9

(5) $81\frac{5}{11}$ Ans. 4.334 +

57. RULE. Extract the required root of numerator and denominator separately; or, reduce the fraction to a decimal, and then extract the required root.

Extraction of Roots other than the Square and Cube Roots.

58. If the index of the root contains no other factors than 2 and 3, we may extract it by means of square and cube root.

59. Let us see why this is so.

$$3 \times 3 \times 3 \times 3 = 81 \text{ or } 3^4 = 81 \text{ hence } \sqrt[4]{81} = 3$$

$$\text{also } \sqrt{81} = 9 \text{ and } \sqrt{9} = 3$$

From this we see that the fourth root of a number equals the square root of the square root of that number.

$$\text{Now } 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729 \text{ or } 3^6 = 729 \text{ hence } \sqrt[6]{729} = 3$$

$$\text{also } \sqrt[3]{729} = 9 \text{ and } \sqrt{9} = 3 \text{ or } \sqrt{729} = 27 \text{ and } \sqrt[3]{27} = 3$$

From this we see that the sixth root equals the square root of the cube root, or the cube root of the square root of a number.

Therefore we may consider the roots as being made up of factors.

60. RULE. Separate the index of the root into its prime factors (we can consider only 2's and 3's), and extract in succession the roots indicated by the several factors. The final result will be the required root.

EXAMPLES FOR PRACTICE.

(1) Find the fourth root of 2401 Ans. 7

(2) Find the fourth root of 208.5136 Ans. 3.8

(3) Find the sixth root of 531441 Ans. 9

RATIO.

61. **Ratio** is the relation with respect to magnitude, which a quantity or number bears to another quantity or number of the same kind. In other words it is the number of times one quantity is contained in another or the quotient obtained by dividing the one by the other. The ratio of 15 to 3 is 5 because 3 is contained in 15 five times. The ratio of 3 to 15 is $\frac{1}{5}$ because 3 is $\frac{1}{5}$ of 15.

62. The two numbers given are called the Terms of the Ratio. The preceding one is called the **Antecedent**, the succeeding, the **Consequent.** The antecedent is the dividend, the consequent the divisor. The value of the ratio is the quotient resulting from the division. The *symbol* of ratio is two dots (:). Thus we write the ratio of 7 to 11 in this manner 7 : 11. We might also express the relation thus, $\frac{7}{11}$.

63. A ratio like the above is called a **Direct Ratio.** In it the antecedent is divided by the consequent, such as the ratio of 7 to 11 is 7 : 11 or $\frac{7}{11}$. If, however, we say the **Inverse Ratio** of 7 to 11, we mean $\frac{11}{7}$, *i. e.*, the consequent divided by the antecedent.

64. As the ratio is always an abstract number we can have a ratio between similar quantities only. Thus we have a ratio between 75 rivets and 16 rivets, but no ratio between 75 rivets and 16 horses.

65. Ratio is not changed by dividing or multiplying both of the terms by the same number, for we are simply dividing or multiplying both terms of a fraction by the same number, which is equivalent to multiplying or dividing the fraction by one, which produces no change. Thus if we have 16 : 32 we may divide each term by 16 and have 1 : 2, or we may multiply each by 2 obtaining 32 : 64 for $\frac{16}{32} = \frac{1}{2} = \frac{32}{64}$.

66. If we use the term **Vary** we imply a ratio. If we say that two numbers vary as two other numbers we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

EXAMPLES FOR PRACTICE.

What is the value of each of the following:

(1)	51 : 17	Ans. 3
(2)	6.6 : 1.32	Ans. 5
(3)	\$16 : \$48	Ans. $\frac{1}{3}$
(4)	$7\frac{1}{4} : 8\frac{1}{2}$	Ans. $\frac{29}{34}$
(5)	What is the inverse ratio of 9 : 81	Ans. 9
(6)	What is the inverse ratio of 86 : $21\frac{1}{2}$	Ans. $\frac{1}{4}$
(7)	What is the inverse ratio of 1.3 : 5.2	Ans. 4

PROPORTION.

67. A proportion is an expression of equality between equal ratios. We may indicate a proportion by the sign : :, or the sign =. For example, to express the proportion 3 is to 9 as 6 is to 18 we may write:

$$3 : 9 : : 6 : 18$$
$$\text{or } 3 : 9 = 6 : 18$$

which is the same as writing $\frac{3}{9} = \frac{6}{18}$.

68. Each antecedent and consequent form a couplet. Take 3 : 9 : : 6 : 18. Here 3 and 9 form a couplet, and 6 and 18 make another couplet, since 3 and 9 are compared, and 6 and 18 compared. The ratio in each case is, of course, the same.

69. The terms of a proportion are numbered. Take 3 : 9 : : 6 : 18. They are designated as in the following proportion:

first	*second*		*third*	*fourth*
3	: 9	: :	6	: 18

The first and fourth terms are called the **Extremes.** The second and third the **Means.** Thus 3 and 18 are the extremes, and 9 and 6 the means.

70. A **Direct Proportion** is one in which both couplets are *direct* ratios. An **Inverse Proportion** is one in which one of the couplets is an *inverse* ratio.

71. Let us now observe the following proportion:

$$3 : 9 : : 6 : 18$$
$$9 \times 6 = 54$$
$$3 \times 18 = 54$$

Thus we see that *the product of the means equals the product of the extremes.*

72. We may express this in another way, as follows:

$$\frac{6 \times 9}{3 \times 18} = 1 \text{ for both } 6 \times 9 \text{ and } 3 \times 18 \text{ equal } 54.$$

or, for the same reason, we may say,

$$\frac{3 \times 18}{6 \times 9} = 1$$

From these expressions we may obtain the following:

$$\frac{6 \times 9}{3} = 18 \text{ and } \frac{6 \times 9}{18} = 3$$

$$\text{also } \frac{3 \times 18}{6} = 9 \text{ and } \frac{3 \times 18}{9} = 6$$

Hence, if any term of a proportion is lacking, we may obtain it from the following conclusions (drawn from the preceding.)

1. The product of the means divided by either extreme gives the other extreme.

2. The product of the extremes divided by either mean gives the other mean.

73. In stating a proportion where one term is unknown, place an x in its place, and then find out what x should be.
As examples:

$$18 : 6 : : 9 : x$$

$$x = \frac{9 \times 6}{18} = 3$$

Hence, the missing term is 3.

$$6 : 18 : : 3 : x$$

$$x = \frac{3 \times 18}{6} = 9 \qquad \text{Ans. } x = 9.$$

$$9 : 3 : : 18 : x$$

$$x = \frac{18 \times 3}{9} = 6 \qquad \text{Ans. } x = 6.$$

$$3 : 9 : : 6 : x$$

$$x = \frac{6 \times 9}{3} = 18 \qquad \text{Ans. } x = 18.$$

74. We may extract any root of *both* sides of a proportion, or we may raise *both* sides to any power without destroying the proportion. To illustrate this in the case of extracting a root, consider the proportion:

$$1 : 9 = 49 : 441$$

Taking the square root of each term we have

$$1 : 3 = 7 : 21$$

$$\text{or } \tfrac{1}{3} = \tfrac{7}{21}$$

and this equality being true shows that the proportion is not destroyed.

In the case of raising to a power consider this proportion:

$$1 : 2 = 2 : 4$$

By cubing each term we obtain

$$1 : 8 = 8 : 64$$

or $\frac{1}{8} = \frac{8}{64}$ and since this equality is true, the proportion has not been destroyed.

75. If the proportion is an inverse one, it is formed in the same manner as if direct except that we *invert one couplet.*

76. Let us now take a case of inverse proportion:

A train running 27 miles per hour covers a certain distance (189 miles) in 7 hours. How long would it take a train running at 32 miles per hour?

Now if we used a direct proportion we would have $27 : 32 :: 7 : x$, but in this case x would equal $8\frac{3}{27}$ hours, or the train which ran 32 miles per hour would take longer to cover a certain distance than one which ran only 27 miles per hour. Obviously, this is not so. Thus we must have an inverse proportion. This would be (inverting one couplet),

$$27 : 32 :: x : 7$$

or $$32 : 27 :: 7 : x$$

Either of these (or any other inverse proportion formed by them) gives

$$x = \frac{27 \times 7}{32} = 5\tfrac{29}{32} \text{ hours. Ans.}$$

77. When we have an example, we should always see *what is to be found* and go to work accordingly. For example:

If a joint 16 feet long requires 80 rivets, what number will a joint 11 feet long require?

Here we wish to find the *number of rivets required*:

16 ft. : 11 ft. : : 80 rivets : number of rivets required.

$$x = \frac{80 \times 11}{16} = 55.$$

Hence, 55 rivets are required.

78. **Simple Proportion.** The above are illustrations of examples in *simple* proportion, or where the proportion is formed by four quantities.

79. To solve examples by simple proportion:

1. Make the number that is of the same denomination as the required answer, the third term. Make x the fourth term.

2. Determine by the nature of the example whether the answer is to be *greater* or *less* than the third term; if it is to be *greater*, place the *greater* of the other two numbers for the second term, but if it is to be *less*, place the smaller for the second term and the other for the first.

3. Multiply the third term by the second term and divide the product by the first term. The result will be the value of x.

80. To illustrate the use of the rule take the following examples:

How many feet of plank will be required for a bridge 530 feet long, if 18,800 feet of plank are required for one 300 feet long?

$$300 : 530 : : 18{,}800 : x$$

$$x = \frac{18{,}800 \times 530}{300}$$

$$x = 33{,}213\tfrac{1}{3} \text{ feet. Ans.}$$

The number that is of the same denomination as the required answer is 18,800, which we place for the third term; x is made the fourth term.

It is evident that *more* feet of plank will be required for a bridge 530 feet long than for one 300 feet long, therefore, the answer will be *greater* than the third term. Hence, 530 is the second term and 300 the first.

Following the rule the above result is obtained.

If a wheel that is $8\frac{1}{2}$ feet in circumference turns 4,250 times on a journey, how many times will a wheel that is 10 feet in circumference turn on the same journey?

$$10 : 8.5 : : 4{,}250 : x$$

$$x = \frac{4{,}250 \times 8.5}{10}$$

$$x = 3{,}612.5 \text{ times. Ans.}$$

The number which is of the same denomination as the required answer is 4,250, which is made the third term; x is the fourth term.

A wheel 10 feet in circumference will not turn as many times on the journey as one which measures $8\frac{1}{2}$ feet. It is apparent that the answer will be less than the third term, therefore, $8\frac{1}{2}$ is placed as the second term and 10 as the first.

EXAMPLES FOR PRACTICE.

Find the value of x in each of the following:

(1) $7 : 36 :: 56 : x$ Ans. 288

(2) $12 : x :: 1 : 144$ Ans. 1728

(3) $\$175.35 : x :: \frac{1}{8} : \frac{3}{7}$ Ans. $\$601.20$

(4) $\frac{5}{16} : \frac{3}{8} :: x : \frac{2}{5}$ Ans. $\frac{1}{3}$

(5) If 4 tons of coal cost $\$24$, what will 18 tons cost at the same rate? Ans. $\$108$.

(6) How many men will be required to build 32 rods of wall in the same time that 5 men can build 10 rods? Ans. 16 men.

(7) If a railroad train runs 30 miles in 50 minutes, in what time will it run 260 miles? Ans. 7 hours $13\frac{1}{3}$ minutes

(8) If 15 tons of lead cost $\$105$, what will 765 tons cost?

Ans. $\$5,355$.

SECTION 4.

ELEMENTARY ALGEBRA.

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1. Algebra is a branch of mathematics in which **quantities** are represented by **letters** and *processes* by *signs*. It may be considered an extension of Arithmetic.

SYMBOLS.

The quantities and processes used in algebra are represented by symbols.

2. **Symbols of Quantity** are the letters of the alphabet and the figures used in Arithmetic.

The first letters of the alphabet and figures are used to represent those quantities whose values are *known*.

Unknown Quantities are those whose values are to be found and are represented by the last letters of the alphabet.

3. **Symbols of Operation** indicate that some process is to be performed.

4. The sign of *addition* $+$ is called **plus**; thus, $8+4$ is read "8 plus 4" and means that 4 is to be added to 8. Similarly, $a+b$ means that the quantity represented by b is to be added to the quantity represented by a.

5. The sign of *subtraction* $-$ is called **minus**; thus, $e-f$ is read "e minus f" and means that f is to be subtracted from e.

6. *Multiplication* is expressed by the sign $\times$ which is read "multiplied by" or "times."

7. *Division* may be expressed by the sign $\div$ or by writing the dividend above and the divisor below a horizontal line. Thus,

$$a \times c \div (e+f) \text{ is the same as } \frac{a \times c}{e+f}.$$

8. These signs are the same as those used in Arithmetic. In multiplication, however, there is a difference. In Arithmetic,

the sign $\times$ must *always* be used, but in Algebra it is frequently omitted.

Thus,

$$\frac{a\,b\,m\,y}{x\,y} \text{ is the same as } \frac{a \times b \times m \times y}{x \times y}.$$

If, however, we omit the sign between the figures in the expression, $\frac{2 \times 3 \times 16}{2 \times 3}$, the value is entirely different.

$$\frac{2 \times 3 \times 16}{2 \times 3} = 16 \qquad \frac{2316}{23} = 100.7 \text{ (nearly)}.$$

We can write $6 \times a \times b$ as $6\,a\,b$. Also, $12 \times 4 \times m \times n$ as $12 \times 4\,m\,n$ but not $124\,m\,n$.

9. We may say: *The sign of multiplication may be omitted between letters and between letters and figures but* **not between figures.** When there is no sign between letters multiplication is understood. For this reason the other signs *must* be used.

The only exception to the above statement is found in the case of the $+$ sign; it may be omitted at the beginning. Thus, $2\,a + 3\,b$ is the same as $+2\,a + 3\,b$. The sign of $2\,a$ is understood to be $+$.

10. Sometimes a point or period is used in place of the multiplication sign; thus,

$$a.\,b.\,c.\,d. = a \times b \times c \times d = a\,b\,c\,d.$$

11. A *Coefficient* is a number prefixed or placed before a quantity to indicate how many times the quantity is to be taken. Thus, in $3\,a\,b$, 3 is the coefficient and shows that $a\,b$ is to be taken 3 times. In other words $3\,a\,b$ is the same as $a\,b + a\,b + a\,b$.

Let us consider the expression $6\,m\,n$. The coefficient of $m\,n$ is 6 and $6\,m$ is considered the coefficient of n. Similarly, $6\,n$ is the coefficient of m.

When no coefficient is expressed, it is understood to be 1. Thus c is the same as $1\,c$ because c multiplied by 1 is c.

12. An *Exponent* is a figure or letter written at the right and a little above a quantity to indicate the number of times the quantity is to be taken as a *factor*. For example, in c^4 the ^{4} shows that c is to be taken 4 times as a factor. c^4 is equivalent to $c \times c \times c \times c$.

13. When no exponent is written, the *first power* is understood; b is the same as b^1.

b^2 is "b to the second power" or b square; it is equal to $b\,b$.

b^3 is "b to the third power" or b cube; it is equal to $b\,b\,b$.

b^4 is "b to the fourth power" and is equal to $b\,b\,b\,b$.

The distinction between coefficient and exponent must be thoroughly understood. In the case of the coefficient, the coefficient shows how many times the quantity is taken to make a given *sum*. The exponent indicates how many times the quantity is to be taken to make a given *product*. Thus,

$$4a = a + a + a + a.$$
$$a^4 = a \times a \times a \times a.$$

14. Symbols of Relation are the signs which show equality and inequality.

The sign $=$ is read "equals." The expression $a = b$ means that the quantity represented by a is equal to the quantity represented by b.

When two quantities are connected by the $=$ sign, the expression is called an *equation*.

The sign $>$ is read "is greater than" and the sign $<$ "is less than." Thus, $a > b$ means that the quantity represented by a is greater than the quantity represented by b. $m < n$ means that m is less than n.

The sign $\backsimeq$ means "is equivalent to."

15. Symbols of Abbreviation. The following are the most common symbols of abbreviation:

Radical Sign	$\sqrt{}$	Braces	$\{\ \}$
Parentheses	$(\ )$		
Brackets	$[\ ]$	Vinculum	$\overline{}$

16. We learned the meaning of the $\sqrt{}$ when studying Arithmetic.

In Algebra the parentheses, brackets, braces and vinculum are used in the same way as in Arithmetic and have exactly the same meaning.

If several quantities are enclosed in parentheses, brackets, etc., they are to be taken as a *single quantity*. For example,

$$7 - 3 + (8 - 2).$$

This expression means that 3 is to be subtracted from 7 and the result thus obtained added to the difference between 8 and 2.

$$7 - 3 + (8 - 2) = ?$$
$$7 - 3 + 6 = 10$$

Suppose we have $12 - 6 - (3 - 1) + 22 - (12 + 1)$.

We may first simplify the quantities in parentheses. $(3 - 1) = 2$, and $(12 + 1) = 13$. Then we may write:

$$12 - 6 - (3 - 1) + 22 - (12 + 1) =$$
$$12 - 6 - 2 + 22 - 13 = 13$$

17. The same rules apply to letters. For example, $4 + a + (b - c)$ means that the difference between b and c is to be added to the sum of 4 and a.

We know that when there is no sign between letters they are to be multiplied. Thus $6\ a\ (x + y)$ means that $x + y$ is to be multiplied by $6\ a$, the result being $6\ a\ x + 6\ a\ y$. In other words both x and y are to be multiplied by the $6\ a$.

18. Suppose we wish to add 7 and $10\ a$ and subtract $3\ b$; we can write it in any of the following ways:

$$(7 + 10\ a - 3\ b)$$
$$[7 + 10\ a - 3\ b]$$
$$\{7 + 10\ a - 3\ b\}$$
$$\overline{7 + 10\ a - 3b.}$$

The expression $4 \times \overline{3 + 4 + 2}$ is equivalent to $4 \times 9 = 36$. The vinculum is most used with the root signs, thus $\sqrt[3]{9 \times 3\ a \times b}$ means the cube root of $27\ a\ b$. The expression $\sqrt{a + b}$ indicates the square root of the expression $a + b$, not $\sqrt{a} + \sqrt{b}$. Let us note the difference; let $a = 9$ and $b = 16$. Then,

$$\sqrt{a + b} = \sqrt{9 + 16} = \sqrt{25} = 5$$
$$\sqrt{a} + \sqrt{b} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

EXAMPLES FOR PRACTICE.

SIMPLIFY,

1. $(14 + 13) \times (5 - 2) = ?$ 81 Ans.
2. $25 - (6 + 7) = ?$ 12 Ans.
3. $(15 - 5) \div 2\,a = ?$ $\frac{5}{a}$ Ans.
4. $(18 - 3 + 6 + 7) \div (11 - 4) = ?$ 4 Ans.
5. $(12 + (\overline{7 - 3) \div 2}) \times 6\,m = ?$ 84 m Ans.

ALGEBRAIC EXPRESSIONS.

An algebraic expression is a combination of letters and signs; such as $2\,a + a\,x - b$.

19. A *term* is an algebraic expression none of whose parts are separated by the + or − signs. In the above expression, $2\,a$, $a\,x$ and b are terms.

Positive terms are those preceded by a + sign. If no sign is expressed the plus sign is understood and the term is positive. Thus in the above expression $2\,a$ and $a\,x$ are positive terms. $6\,l\,m$ is a positive term.

Negative terms are those preceded by a − sign. Thus in the above expression $-b$ is a negative term. *This sign must never be omitted.* Equal terms with unlike signs cancel each other; thus

$$2\,c - 2\,c = 0.$$

20. A **monomial** is an expression containing but *one* term, as $7\,d$. If there are two terms the expression is called a **binomial.** Expressions of more than one term are called **polynomials.** Thus $x + 3\,b - y$ is a polynomial.

21. **Similar terms,** sometimes called like terms, are those differing only in numerical coefficient. Thus $5\,a\,b$, $8\,a\,b$ and $3\,a\,b$ are similar terms. $4\,a\,b$ and $7\,b$ are *unlike terms.*

NEGATIVE QUANTITIES.

22. The signs + and − are used to indicate addition and subtraction. They are also used to distinguish between quantities opposite in character. Plus quantities are those greater than zero and minus quantities those less than zero. A thermometer

shows + and — temperatures or temperatures above and below zero.

A man's property may be called + and his debts — . If a man owes $100 we say that his property is — $100; if he has $75 his property may be expressed as + $75 or simply $75.

If a thermometer inside a house indicates 60° and one outside indicates 10° below zero, the difference is 70° because the temperature outside must first rise 10° to become zero and then 60° more, making a total of 70°.

By means of the same reasoning, we see that a man that has $50 is $100 better off than the man who owes $50.

ADDITION.

23. Addition is the process of finding the aggregate or sum of two or more quantities. Thus the sum of 4 and 6 is 10. The sum of 5 a and 6 a is 11 a.

If we add 5 bolts, 8 bolts, 9 bolts and 3 bolts, the sum is 25 bolts. If we let b represent one bolt we add 5 b, 8 b, 9 b and 3 b. The result is 25 b.

In the same way $-3a-6a-9a$ and $-13a$ are equal to $-31a$.

24. When the terms are similar and the signs alike, *add the coefficients and to the sum annex the letter or letters; the sign of the sum will be the common sign.*

25. If the terms do not have the same signs but are similar *add together all the + quantities and all the — quantities and find the difference. The sign of the result will be the sign of the greater sum.*

$13\,a\,c^2 + 6\,a\,c^2 - a\,c^2 + 4\,a\,c^2 - 8\,a\,c^2 = 23\,a\,c^2 - 9\,a\,c^2 = 14\,a\,c^2$.

(1)	(2)	(3)	(4)	(5)
$5\,a\,x$	$-3\,x^2$	$+12\,x\,y$	$-4\,c$	$-6(x+y)$
$3\,a\,x$	$-7\,x^2$	$-17\,x\,y$	$+3\,c$	$9(x+y)$
$8\,a\,x$	$-x^2$	$+13\,x\,y$	$-9\,c$	$-2(x+y)$
$2\,a\,x$	$-4\,x^2$	$+7\,x\,y$	$-6\,c$	$(x+y)$
$a\,x$	$-6\,x^2$	$-6\,x\,y$	$+4\,c$	$-13(x+y)$
$19\,a\,x$	$-21\,x^2$	$+9\,x\,y$	$-12\,c$	$-11(x+y)$

26. The sum of a and 2 a is 3 a, but the sum of 3 a and 7 c

is neither 10 a nor 10 c but is $3a + 7c$. Hence if the terms are unlike they cannot be added; *the addition is indicated by connecting them with the proper sign.*

27. Suppose we wish to add several terms. For instance, $8am + 7cm - 2\sqrt{x} - x - 3am + 4x + 5\sqrt{x} - 3am + cm - 3x - \sqrt{x} - am$. We can write similar terms under each other and add them using the principles already learned.

$$\begin{array}{r} 8am + 7cm - 2\sqrt{x} - x \\ -3am + cm + 5\sqrt{x} + 4x \\ -3am - \sqrt{x} - 3x \\ -am \\ \hline am + 8cm + 2\sqrt{x} \end{array}$$

(1)

$$\begin{array}{r} 7x - 3y \\ -3x + y \\ 2x + 6y \\ \hline 6x + 4y \end{array}$$

(2)

$$\begin{array}{r} 5a + 3c - 4 \\ -7a - c + 12 \\ 4a - 6c \\ \hline 2a - 4c + 8 \end{array}$$

(3)

$$\begin{array}{r} 3ab + 5c \\ 8ab + 2c \\ -3ab \\ \hline 8ab + 7c \end{array}$$

EXAMPLES FOR PRACTICE.

1. Add together $3ax - 4ab - xy + z - 3xy + 6ab + 7xy - 3z - 4ax - 6xy + ax$.

Ans. $2ab - 3xy - 2z$.

2. Add together $17ax^2 - 9axy + 6z - 24 + 6 + 2axy - 2z + ax^2 - 13 - 14ax^2 + 6axy - 4z$.

Ans. $4ax^2 - axy - 31$.

SUBTRACTION.

28. Subtraction is the process of taking one quantity from another, or it is finding the difference between two quantities.

29. The **Subtrahend** is the quantity to be subtracted.

30. The **Minuend** is the quantity from which the subtrahend is to be subtracted.

31. The **Remainder** is the result.

Let us subtract $14a$ from $26a$.

$$\begin{array}{ll} 26a & \textit{Minuend.} \\ 14a & \textit{Subtrahend.} \\ \hline 12a & \textit{Remainder.} \end{array}$$

This shows that the minuend is equal to the sum of the subtrahend and remainder.

We could write the above as follows:

$$26\,a - 14\,a = 12\,a.$$

32. In a certain time A made 24 dollars and B lost 8 dollars. What was the difference in the profits of A and B. Gain or profit is considered +, and loss —. To find the difference we must take — 8 dollars from 24 dollars.

$$24 \text{ dollars} - (-8 \text{ dollars}) = 32 \text{ dollars}.$$

33. When studying negative quantities (Art. 22), we learned that the difference between a + quantity and a — quantity is equal to the sum of the quantities. The above shows that:

34. *Subtracting a positive quantity is the same as adding an equal negative quantity.*

Art. 32 shows that *subtracting a negative quantity is the same as adding an equal positive quantity.*

To subtract one quantity from another, change the sign of the subtrahend and add the result to the minuend.

Subtract $5\,a$ from $8\,a$.

SUBTRACTING.			ADDING.
$8\,a$	*Minuend*		$8\,a$
$5\,a$	*Subtrahend*	or	$-5\,a$
$3\,a$	*Remainder*		$3\,a$

35. To subtract one polynomial from another *change the sign of each term of the subtrahend and add the result to the minuend.*

From $11\,abc + 3\,x - 7\,y + 48$ take $7\,abc - 6\,x + 9\,y - 76$.

$$\begin{array}{r} 11\,abc + 3\,x - 7\,y + 48 \\ -7\,abc + 6\,x - 9\,y + 76 \\ \hline 4\,abc + 9\,x - 16\,y + 124 \end{array}$$

EXAMPLES FOR PRACTICE.

1. From $-347\,axy$ take $223\,axy$. Ans. $-570\,axy$.
2. Subtract $4\,m^2 - 6\,n^3 + 73\,x$ from $-m^2 - 8\,n^3 + 83\,x$. Ans. $-5\,m^2 - 2\,n^3 + 10\,x$.
3. Take $2\,x^3 - y^2$ from the sum of $x^3 - 2\,xy + 3\,y^2$ and $xy + 4\,y^2$. Ans. $-x^3 - xy + 8\,y^2$.

4. From the sum of $2a^3 - a^2b - 5ab^2$ and $3a^2b - 5ab^2 - 4b^3$ take the sum of $-2a^3 - 7a^2b - 6b^3$ and $-6ab^2 + 5b^3$.

Ans. $4a^3 + 9a^2b - 4ab^2 - 3b^3$.

5. Subtract the sum of $-5x^3 - 12x^2 + 3$ and $-3x^4 + 2x^2 - 5x$ from the sum of $x^4 - 1$ and $2x^3 - 7x - 10x^2$.

Ans. $4x^4 + 7x^3 - 2x - 4$.

USE OF PARENTHESES.

36. The parentheses, brackets, etc., are used so frequently in Algebra and Engineering that their use must be thoroughly understood.

Suppose we have the expression,

$$a - 2m + (3m - n + 4x).$$

We know that it means that the quantity $3m - n + 4x$ is to be *added* to $a - 2m$. Performing the addition we have

$$a - 2m + 3m - n + 4x.$$

37. It will be noted that the signs are the same as though no parenthesis had been used.

The expression,

$$a - 2m - (3m - n + 4x)$$

means that the quantity $3m - n + 4x$ is to be *subtracted* from $a - 2m$. Subtracting $3m - n + 4x$ is the same as adding $-3m + n - 4x$. We have as a result,

$$a - 2m - 3m + n - 4x.$$

38. By comparing this expression with that of Art. 37 we see that when we removed the parenthesis, we simply changed all the signs in the parenthesis. Similarly

$$a^2 - (2ab + c) = a^2 - 2ab - c.$$

39. We can then make the following statements:

If a parenthesis is preceded by a + sign, the parenthesis signs may be removed without changing the + or — signs of the terms within the parenthesis.

If a parenthesis is preceded by a — sign, the parenthesis signs may be removed if the sign of each term within the parenthesis is changed from + to — or from — to +.

40. The same rules apply to brackets, braces, vinculum, etc.

41. After removing the parenthesis similar terms should be united.

Simplify the expression,

$$4\,m^2\,n - (8\,m\,n^2 + 3\,m^2\,n - n^2).$$

$$4\,m^2\,n - 8\,m\,n^2 - 3\,m^2\,n + n^2$$

or, $$m^2\,n - 8\,m\,n^2 + n^2.$$

42. Parentheses are often enclosed by brackets, etc. In such cases the same rules are followed. It is better to remove first the signs from the innermost pair.

$$3\,a - [2\,a - (a + 2)].$$

Removing the parenthesis we have,

$$3\,a - [2\,a - a - 2].$$

Removing the brackets we have,

$$3\,a - 2\,a + a + 2.$$

Combining similar terms, we have,

$$2\,a + 2.$$

EXAMPLES FOR PRACTICE.

Simplify,

1. $x^3 - z^3 - (x^3 + y^3 - z^3)$ Ans. $-y^3$.
2. $a - c + d - (a + c - d) - \overline{d - c - a.}$ Ans. $a - c + d$.
3. $2\,a - \left\{b - [3\,a - (2\,b - a)]\right\}$. Ans. $6\,a - 3\,b$.
4. $5\,x - \Big(4\,x - [-3\,x - \big\{2x - x - 1\big\}]\Big)$ Ans. $1 - 3\,x$.

43. To enclose any terms in parenthesis we observe the same principles.

44. Suppose we wish to enclose in parenthesis the last three terms of the following expression, and have the + sign precede the parenthesis,

$$x^3\,y + x^2\,y^2 - x\,y^3 + y^4.$$

$$x^3\,y + (x^2\,y^2 - x\,y^3 + y^4).$$

If the parenthesis is to be preceded by the — sign, the sign of each term thus enclosed must be changed.

$$x^3\,y + x^2\,y^2 - x\,y^3 + y^4$$

$$x^3\,y - (-x^2\,y^2 + x\,y^3 - y^4).$$

45. Another example. To enclose the last three terms in parenthesis preceded by the — sign.

$$3x - 2y + 5z - 4x$$
$$3x - (2y - 5z + 4x).$$

In the first expression the sign of $2y$ is — and if it is placed within the parenthesis the sign must be changed to +. It is, however, omitted, as the plus sign is understood.

MULTIPLICATION.

46. Multiplication is a method of finding or expressing one quantity as many times as is indicated by the other or others.

In Arithmetic the product of 4×3 is the same as 3×4. Similarly $a \times b = b \times a$.

47. Monomials. First multiply together the *coefficients*; then annex the *letters*, giving to each letter an exponent equal to the *sum* of the exponents of the factors.

48. The sign of the product is + when the signs of the factors are alike and — when the signs are unlike.

49. Multiply $3ab^2$ by $6a^2b$.

Multiplying the coefficients we have 18 for a new coefficient; $a \times a^2 = a^3$ and $b \times b^2 = b^3$.

The product is $18a^3b^3$.

In multiplying the above we proceed as follows: $a \times a^2 = a^3$ because $a^2 = a \times a$ and $a \times a \times a = a^3$. Similarly, $b^2 = b \times b$ and $b \times b \times b = b^3$.

50. *In multiplying letters we add the exponents.* Thus $a^1 \times a^1 = a^2$. Again, $c^3 \times c^5 = c^8$. The same holds good if the exponents are letters. Thus,

$$a^m \times a^m = a^{m+m} = a^{2m}.$$

51. Another example. Multiply $5a^2b^nx$ by $3a^mb^3x^3$.

The new coefficient is $5 \times 3 = 15$.

$$a^2 \times a^m = a^{2+m}$$
$$b^n \times b^3 = b^{3+n}$$
$$x \times x^3 = x^4$$

The product is $15a^{2+m}b^{3+n}x^4$.

52. Multiply $5\,a\,b^2\,c$ by $-2\,b\,c^3$.

The new coefficient will be $5 \times -2 = -10$.

The product of the letters is $a\,b^3\,c^4$ and the result,

$$-10\,a\,b^3\,c^4.$$

Had the sign of the first quantity been $-$ the product would be $10\,a\,b^3\,c^4$, because $-5 \times -2 = +10$.

EXAMPLES FOR PRACTICE.

1. Multiply $8\,x^2\,y$ by $7\,a\,y$. Ans. $56\,a\,x^2\,y^2$.
2. Multiply $-3\,a^3\,b^m$ by $4\,b^n\,c$. Ans. $-12\,a^3\,b^{m+n}\,c$.
3. Multiply $-7\,a^3\,b^x\,c^4$ by $-6\,b^x\,c^e$.

   Ans. $42\,a^3\,b^{2x}\,c^{4+e}$.
4. Multiply $16\,m\,n\,x^3$ by $-a\,b\,c^2$.

   Ans. $-16\,a\,b\,c^2\,m\,n\,x^3$.

53. As any number of terms in a parenthesis is considered as a *monomial* the coefficient of the *parenthesis* is multiplied.

Multiply $4\,(a+b)$ by $3\,a$.

$3\,a \times 4 = 12\,a$ and annexing the quantity in parenthesis we have,

$$12\,a\,(a+b).$$

54. Polynomials. Suppose we wish to multiply $7+2$ by 4. Since multiplication is a short method of finding the sum of the repetitions of a quantity,

$$\begin{array}{l} 7+2 \\ 7+2 \\ 7+2 \\ 7+2 \\ \hline 28+8=36 \end{array} \qquad \text{or} \qquad \begin{array}{l} 7+2 \\ \underline{4} \\ 28+8=36 \end{array}$$

55. Thus we see that we obtain the answer if we multiply *both* terms of the polynomial by the monomial.

Multiply $3\,a+5\,b$ by $8\,x$.

$$\begin{array}{l} 3\,a+5\,b \\ \underline{8\,x} \\ 24\,a\,x+40\,b\,x. \end{array}$$

Multiply $23\,a\,b^2-5\,x\,y+3\,a\,y-1$ by $-7\,a\,x^2\,y$.

$$\begin{array}{l} 23\,a\,b^2-5\,x\,y+3\,a\,y-1 \\ \underline{-7\,a\,x^2\,y} \\ -161\,a^2\,b^2\,x^2\,y+35\,a\,x^3\,y^2-21\,a^2\,x^2\,y^2+7\,a\,x^2\,y. \end{array}$$

56. If both factors, or in other words if both multiplier and multiplicand, are polynominals we proceed in the same way; *multiply each term of the multiplicand by each term of the multiplier and add the products.*

In performing multiplication of polynomials the signs are of utmost importance.

57. Example. Multiply $4a - 3b$ by $2a + 7b$.

$$\begin{array}{r} 4a - 3b \\ 2a + 7b \\ \hline 8a^2 - 6ab \\ \quad + 28ab - 21b^2 \\ \hline 8a^2 + 22ab - 21b^2 \end{array}$$

58. Let us take another example. Multiply $4a^2 + 2ab + b^2$ by $4a^2 - 2ab + b^2$.

$$\begin{array}{r} 4a^2 + 2ab + b^2 \\ 4a^2 - 2ab + b^2 \\ \hline 16a^4 + 8a^3b + 4a^2b^2 \\ \quad - 8a^3b - 4a^2b^2 - 2ab^3 \\ \quad + 4a^2b^2 + 2ab^3 + b^4 \\ \hline 16a^4 + 4a^2b^2 + b^4 \end{array}$$

$16a^4 + 4a^2b^2 + b^4$ Ans.

59. If some of the terms are figures the same rules are followed. Multiply $1 - 5x - 3x^2 + 2x^3$ by $3x - 5$.

$$\begin{array}{r} 1 - 5x - 3x^2 + 2x^3 \\ 3x - 5 \\ \hline 3x - 15x^2 - 9x^3 + 6x^4 \\ 25x + 15x^2 - 10x^3 \quad - 5 \\ \hline 28x - 19x^3 + 6x^4 - 5 \end{array}$$

$28x - 19x^3 + 6x^4 - 5$. Ans.

60. In Algebra it is customary to arrange the terms in the answer, and also sometimes in the example, so that the first term will be $+$ and the exponents will be either an ascending or descending scale. Thus in the above problem we should write the answer.

$$6x^4 - 19x^3 + 28x - 5.$$

61. If the product of three factors is to be found, first multiply together two of them and then multiply the product by the third.

62. Example: Multiply together, $a + b$, $a^2 - a\,b + b^2$, $a^3 - b^3$.

$$\begin{array}{r} a^2 - a\,b + b^2 \\ a + b \\ \hline a^3 - a^2\,b + a\,b^2 \\ \quad + a^2\,b - a\,b^2 + b^3 \\ \hline a^3 \qquad\qquad + b^3 \end{array}$$

Now multiply this product by $a^3 - b^3$.

$$\begin{array}{r} a^3 + b^3 \\ a^3 - b^3 \\ \hline a^6 + a^3\,b^3 \\ \quad - a^3\,b^3 - b^6 \\ \hline a^6 \qquad - b^6 \end{array}$$

$a^6 - b^6$. Ans.

63. The product of two or more quantities may be *indicated* by enclosing them in parenthesis. Thus the above example would be written.

$$(a + b)\ (a^2 - a\,b + b^2)\ (a^3 - b^3).$$

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $x + y$ by $x - y$. Ans. $x^2 - y^2$.
2. $2\,x - 3$ by $3\,x + 2$. Ans. $6\,x^2 - 5\,x - 6$.
3. $6\,m - 4$ by $3\,m - 2$. Ans. $18\,m^2 - 24\,m + 8$.
4. $3\,a^2 - a - 2$ by $3\,a + 2$. Ans. $9\,a^3 + 3\,a^2 - 8\,a - 4$.
5. $2\,m^2 + 9\,m - 5$ by $3\,m^2 - 7\,m + 4$.
   Ans. $6\,m^4 + 13\,m^3 - 70\,m^2 + 71\,m - 20$.
6. Find the product
   $(3\,a^2 - a - 2)\ (a + 1)\ (3\,a - 2)$.
   Ans. $9\,a^4 - 13\,a^2 + 4$.

DIVISION.

64. The process of finding one of two factors when the product and one factor are known is called Division. Thus we see that division is the converse of Multiplication.

65. The *product* is called the **Dividend.**

66. The *given factor* is called the **Divisor.**

67. The *result* or the other factor is called the **Quotient.**

68. As in multiplication, like signs give + and unlike signs —.

If the divisor and dividend are both + the quotient is +.
If the divisor and dividend are both — the quotient is +.
If the divisor is + and the dividend — the quotient is —.
If the divisor is — and the dividend + the quotient is —.

69. Divide $16\,a\,b$ by $8\,a$.

$$\frac{16\,a\,b}{8\,a} = 2\,b.$$

Monomials. To find the coefficient of the quotient, divide the coefficient of the dividend by the coefficient of the divisor.

70. To find the exponent of any letter of the quotient, *subtract* the exponent of the letter of the divisor from the exponent of the letter of the dividend.

71. Divide $27\,a^3\,b^3$ by $3\,a^2\,b$.

To find the coefficient of the quotient we divide 27 by 3. $27 \div 3 = 9$.

Now we subtract the exponent of a^2 from that of a^3 and obtain 1 as a result because $3 - 2 = 1$, and any quantity with the exponent 1 is the quantity itself. $a^1 = a$.

Subtracting the exponent of b from that of b^3 we get 2 because $3 - 1 = 2$. Hence the exponent of b should be 2. The entire quotient is $9\,a\,b^2$.

72. In Multiplication we *add* the exponents and in Division we *subtract* them.

Divide a^5 by a^2.

$$\frac{a^5}{a^2} = a^3 \text{ ; because } 5 - 2 = 3.$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

73. Another example. Divide $36\,x^2y\,z^3$ by $-4\,x\,y\,z$.

$$\frac{36\,x^2y\,z^3}{-4\,x\,y\,z} = -9\,x\,z^2.$$

In this case the sign of the quotient is — because the signs of the dividend and divisor are unlike. The exponents of y are the same, hence they cancel.

74. The process of dividing a polynomial by a monomial is very simple. Consider the polynomial made up of several monomials; divide each monomial by the divisor. In doing this pay careful attention to the signs, remembering that like signs produce + and unlike —.

75. Divide $12\,x^4y^6 - 16\,x^5\,y^5 + 20\,x^6\,y^4 - 28\,x^7\,y^8$ by $4\,x^4\,y^3$.

$$4\,x^4\,y^3)\overline{12\,x^4\,y^6 - 16\,x^5\,y^5 + 20\,x^6\,y^4 - 28\,x^7\,y^8}$$
$$3\,y^3 - 4\,x\,y^2 + 5\,x^2\,y - 7\,x^3\,y^5$$

76. Another example: Divide $15\,a^m\,b^n\,c^r - 35\,a^{m+2}\,b^{2n}\,c$ by $5\,a^m\,b^n\,c$.

$$5\,a^m\,b^n\,c)\overline{15\,a^m\,b^n\,c^r - 35\,a^{m+2}\,b^{2n}\,c.}$$
$$3\,c^{r-1} - 7\,a^2\,b^n.$$

EXAMPLES FOR PRACTICE.

1. Divide $20\,x^4y\,z + 15\,x\,y\,z^3 - 10\,x^2\,y$ by $-5\,x\,y$.
   Ans. $-4\,x^3\,z - 3\,z^3 + 2\,x$.
2. Divide $9\,a^4 + 27\,a^3 - 15\,a^2 - 6\,a$ by $3\,a$.
   Ans. $3\,a^3 + 9\,a^2 - 5\,a - 2$.
3. Divide $-75\,m^4n + 30\,m^3n^2 - 15\,m^2n$ by $15\,m^2n$.
   Ans. $-5\,m^2 + 2\,m\,n - 1$.

77. Polynomials. We have seen that the division of monomials is similar to short division in Arithmetic. The division of polynomials is like long division in Arithmetic.

78. Let us take an example. Divide $a^3 - 3\,a^2\,y + 3\,a\,y^2 - y^3$ by $a^2 - 2\,a\,y + y^2$.

First we should write the divisor and dividend as in long division in Arithmetic.

$$a^2 - 2\,a\,y + y^2)\,a^3 - 3\,a^2\,y + 3\,a\,y^2 - y^3\,(\,a - y$$
$$a^3 - 2\,a^2\,y + a\,y^2$$
$$-\,a^2\,y + 2\,a\,y^2 - y^3$$
$$-\,a^2\,y + 2\,a\,y^2 - y^3$$

To find the first term of the quotient divide the a^3 by a^2. $\frac{a^3}{a^2} = a$. Hence a is first term. Now we multiply each term in the divisor by a and after placing the results under similar terms

of the dividend we subtract. We then divide $-a^2y$ by a^2 and get $-y$ for the next term. Multiplying again by $-y$ we find that the terms are the same as those brought down.

79. Let us take another example. Divide $2a^3b - 2ab^3$ by $ab - b^2$.

$$
\begin{array}{l}
ab - b^2 \;)\; 2a^3b - 2ab^3 \;(\; 2a^2 + 2ab \\
\quad\; 2a^3b - 2a^2b^2 \\
\quad\; \overline{\qquad\qquad} \\
\quad\; 2a^2b^2 - 2ab^3 \\
\quad\; 2a^2b^2 - 2ab^3 \\
\quad\; \overline{\qquad\qquad}
\end{array}
$$

As before we divide $2a^3b$ by ab and obtain $2a^2$ for the first term of the quotient. Now multiplying both terms of the divisor by $2a^2$ we get $2a^3b - 2a^2b^2$. The $2a^3b$ cancels but we have no quantity similar to $-2a^2b^2$ so we bring it down *with the sign changed*. The new dividend is $2a^2b^2 - 2ab^3$ and we find that the divisor is contained in it $2ab$ times.

80. Divide $m^4 - 1$ by $m - 1$.

$$
\begin{array}{l}
m - 1 \;)\; m^4 - 1 \;(\; m^3 + m^2 + m + 1. \text{ Ans.} \\
\quad\; m^4 - m^3 \\
\quad\; \overline{\qquad\qquad} \\
\quad\quad + m^3 - 1 \\
\quad\quad + m^3 - m^2 \\
\quad\quad \overline{\qquad\qquad} \\
\quad\quad\quad + m^2 - 1 \\
\quad\quad\quad + m^2 - m \\
\quad\quad\quad \overline{\qquad\qquad} \\
\quad\quad\quad\quad + m - 1 \\
\quad\quad\quad\quad + m - 1 \\
\quad\quad\quad\quad \overline{\qquad\qquad}
\end{array}
$$

If the dividend is not exactly divisable by the divisor, the remainder should be placed over the divisor and connected to the quotient by the proper sign.

EXAMPLES FOR PRACTICE.

1. Divide $a^4 + 4c^4$ by $a^2 - 2ac + 2c^2$.

   Ans. $a^2 + 2ac + 2c^2$.

2. Divide $x^4 + 4x^2y^2 + 3y^4$ by $x + 2y$.

   Ans. $x^3 - 2x^2y + 8xy^2 - 16y^3 + \dfrac{35y^4}{x + 2y}$.

3. Divide $10a^3 - 20a^2x + 30x^3$ by $a + x$.

   Ans. $10a^2 - 30ax + 30x^2$.

4. Divide $2c^4 - 3c^3 - 12 - 4c^2 - 11c$ by $2c^2 + c + 4$.

   Ans. $c^2 - 2c - 3$.

CANCELLATION.

As shown in Arithmetic, cancellation often shortens the work and a few rules concerning its use will now be given.

81. CASE I. Suppose we have the expression

$$\frac{18 \times 6 \times 10 \times 2}{24 \times 3} = ?$$

We can perform the cancellation thus,

$$\frac{\overset{6}{\cancel{18}} \times \overset{1}{\cancel{6}} \times \overset{5}{\cancel{10}} \times \overset{1}{\cancel{2}}}{\underset{4}{\underset{2}{\cancel{24}}} \times \underset{1}{\cancel{3}}} = 30$$

82. If the signs in the numerator had been + instead of × we could not have cancelled as above. When we cancel we simply divide out factors. Now if these quantities are connected by the plus sign, we cannot cancel out a factor unless it is contained a whole number of times *in every quantity so connected.*

For instance let us take the expression:

$$\frac{18 + 6 + 10 + 2}{24 \times 3} = ?$$

83. In the numerator, 2 is contained a whole number of times in each quantity. Also, 2 is a factor of one quantity, (24), of the denominator. Hence we divide *each* quantity of the numerator and one quantity of the denominator by 2. Thus,

$$\frac{\overset{9}{\cancel{18}} + \overset{3}{\cancel{6}} + \overset{5}{\cancel{10}} + \overset{1}{\cancel{2}}}{\underset{12}{\cancel{24}} \times 3} = \frac{9 + 3 + 5 + 1}{12 \times 3} = \frac{18}{36} = \frac{1}{2}$$

84. Hence we may say, when the quantities, (letters or numbers) of the numerator are connected by the + (or —) sign, and those in the denominator by the × sign: *Cancel the same factor from each quantity of the numerator as from one quantity of the denominator.*

85. CASE II. Suppose we have the expression,

$$\frac{18 \times 6 \times 10 \times 2}{24 + 3} = ?$$

We see that this case is exactly the reverse of the preceding.

The factor 3 is contained in both 24 and 3 of the denominator and in 18 of the numerator. We cancel as follows:

$$\frac{\overset{6}{\cancel{18}} \times 6 \times 10 \times 2}{\underset{8}{\cancel{24}} + \underset{1}{\cancel{3}}} = \frac{6 \times 6 \times 10 \times 2}{8 + 1} = \frac{720}{9} = 80.$$

86. In regard to this case we may say; when quantities in the numerator are connected by the $\times$ sign and those of the denominator by the $+$ or $-$ sign: *Cancel the same factor from each quantity in the denominator as from one quantity of the numerator.*

87. Case III. Again, suppose all the signs are either $+$ or $-$

$$\frac{6 - 9 + 81 - 3}{27 + 18} = ?$$

First, we must see if there is some factor contained in each quantity. We see that 3 is the factor. Then we can cancel as follows:

$$\frac{\overset{2}{\cancel{6}} - \overset{3}{\cancel{9}} + \overset{27}{\cancel{81}} - \overset{1}{\cancel{3}}}{\underset{9}{\cancel{27}} + \underset{6}{\cancel{18}}} = \frac{2 - 3 + 27 - 1}{9 + 6} = \frac{25}{15} = \frac{5}{3}$$

88. When the quantities in both the numerator and denominator are connected by the $+$ (or $-$) sign: *Cancel the factor or factors which are contained in every quantity.*

89. We may have examples containing both letters and figures but the principles governing the cancellation are the same, that is, if letters occur in both the numerator and the denominator they may be cancelled in the same manner as figures.

Let us consider the expression,

$$\frac{16 + 3 + 2 - 15}{3 + 2 + 5} = ?$$

90. We cannot cancel because there is no factor (except 1) which is contained without a remainder in every quantity. However, in the following expression cancellation greatly simplifies the work.

$$\frac{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{22}}_{2} \times \overset{1}{\cancel{40}}_{3}}{\underset{1}{\cancel{11}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{2}} \times \underset{2}{\cancel{16}}_{1} \times \underset{1}{\cancel{3}}} = \frac{1}{1} = 1$$

EQUATIONS.

91. When two expressions are equal to each other, they may be written with the sign of equality between them. The whole expression is called an **equation.** Thus,

$$a + 2\,b + x = m + 4\,n.$$

92. The *first member* is the expression on the left of the sign of equality and the *second member* is the expression on the right. The members are also called *sides.*

93. In the above equation $a + 2\,b + x$ is the first member, and $m + 4\,n$ the second. If we should change any quantity such as a to the *other side* it would mean that a is written on the right-hand side instead of on the left-hand side.

94. Since an equation is an expression of **equality,** the sum of the quantities on the left-hand side of the sign of equality is equal to the sum of those on the right-hand side. In other words, the first member is equal to the second.

In the equation,

$$a + 2\,b = m + n$$

let $a = 3$, $b = 2$, $m = 6$ and $n = 1$.

Substituting these values we have,

$$\begin{aligned} a + 2\,b &= m + n \\ 3 + (2 \times 2) &= 6 + 1 \\ 3 + 4 &= 6 + 1 \\ 7 &= 7 \end{aligned}$$

Suppose now we add 8 to each side of the equation; the result is,

$$\begin{aligned} a + 2\,b + 8 &= m + n + 8 \\ 3 + 4 + 8 &= 6 + 1 + 8 \\ 15 &= 15 \end{aligned}$$

We see that the first member is still equal to the second. Now let us divide each side by a which equals 3.

$$\frac{a + 2\,b + 8}{a} = \frac{m + n + 8}{a}$$

$$\frac{3 + 4 + 8}{3} = \frac{6 + 1 + 8}{3}$$

$$\frac{15}{3} = \frac{15}{3} \text{ or } 5 = 5$$

95. Thus we see that *if the same quantity is* **added** *to both sides, or if both sides are* **divided** *by the same quantity, the first member is still equal to the second.* Similarly, it can be shown that if the same quantity is **subtracted** from *both sides* or if both sides are **multiplied** by the same quantity, the left side of the equation is equal to the right.

96. Suppose, $a = 6$, $b = 4$ and $c = 2$, and we wish to find the value of the expression,

$$a + b - 3\ c.$$

In such cases we write it as an equation with the unknown quantity represented by x. Thus,

$$x = a + b - 3\ c.$$

The expression means that x is equal to $a + b - 3\ c$.

Then,

$$x = a + b - 3\ c.$$
$$x = 6 + 4 - (3 \times 2)$$
$$x = 6 + 4 - 6$$
$$x = 4$$

We have found the value of x or in other words we have *solved the equation.*

97. Another example: Find the value of x from the equation.

$$x = \frac{6\ a + 5\ c}{2\ b}, \text{ if } a = 3, b = 5, \text{ and } c = 2.$$

Replacing a, b and c, by their respective values,

$$x = \frac{(6 \times 3) + (5 \times 2)}{2 \times 5}$$
$$x = \frac{18 + 10}{10}$$
$$x = 2.8 \text{ Ans.}$$

98. Suppose an engine makes 125 revolutions per minute and develops .4 horse-power per revolution, what is the total horse-power?

As the total horse-power is the unknown quantity, we will let it be represented by x. The known quantities, 125 revolutions and .4 horse-power, can be represented by a and b respectively. Thus,

$a = 125$ revolutions. $b = .4$ horse-power.

$x =$ the total horse-power.

We know that the total horse-power is equal to the number of revolutions multiplied by the horse-power per revolution, or,

$$x = a \times b.$$
$$x = 125 \times .4.$$
$$x = 50 \text{ horse-power.} \quad \text{Ans.}$$

99. Equations are not always written with x representing the unknown quantity, any other letter may be used.

TRANSPOSING.

100. There are many equations in which the unknown quantity is not alone on the left-hand side; it frequently happens that it is among several known quantities on the right-hand side. In such cases it is usual to *rewrite* the equation so that the unknown quantity will be alone on the left-hand side. To do this quantities must be **transposed** from one side to the other.

101. Let us consider this equation,

$$a = c - x.$$

If we subtract a from both sides we will have,

$$a - a = c - a - x.$$

Now let us add x to both sides,

$$a - a + x = c - a - x + x.$$

We know that $a - a = 0$ and $- x$ and $+ x$ cancel, hence we may write;

$$x = c - a.$$

102. In brief, we have *transposed* the x to the left-hand side, *transposed* the a to the right-hand side (because a known quantity) and let the c *remain on the right-hand side* (because a known quantity).

103. It will be noted that in changing the x to the other side, the **sign is changed.** Also the sign of the a is now $-$ instead of $+$. The sign before c is the same as before.

104. Any term may be transposed from one side to the other by changing the sign.

105. Let us consider this equation,

$$5x + 3ab = 2x + 3ab + m.$$

Here we have terms containing x on both sides. First we place the $2x$ on the left-hand side *with its sign changed.*

$$5x + 3ab - 2x = 3ab + m.$$

Next we write the $3ab$ on the right-hand side *with its sign changed.*

$$5x - 2x = 3ab - 3ab + m.$$

Combining like terms,

$$3x = m.$$

Since $3x = m$, one x equals $m \div 3$ or $x = \frac{m}{3}$

106. In other words *to find the value of x we divide by the* **coefficient** *of x.*

NOTE. **In solving problems, be careful to make a distinction between the letter x and the multiplication sign $\times$.**

EXAMPLES FOR PRACTICE.

1. Find the value of x in the equation, $8x - 12 = 6x + 4$.

   Ans. $x = 8$.

2. Solve the equation, $37x - (4 + 7) = 41x + 25$.

   Ans. $x = -9$.

3. Find the value of x in the equation,

   $2ax + c = 3c - 5ax + 26d$

   $a = 3$, $c = 8$ and $d = 1$. Ans. $x = 2$.

4. Solve the equation,

   $7x - ab + c = 3x + 12$.

   $a = 5$, $b = 3$ and $c = 7$. Ans. $x = 5$.

107. Let us consider another equation:

$$\frac{x}{a} + c = a^2 - b^2.$$

First we transpose c to the right-hand side.

$$\frac{x}{a} = a^2 - b^2 - c.$$

To obtain x alone, we must multiply both sides by a.

$$\frac{x}{a} \times a = a(a^2 - b^2 - c).$$

or, $$x = a^3 - ab^2 - ac.$$

108. The left-hand side of the equation shows that multiplying a fraction by the *denominator* gives the numerator as a result, that is, $\frac{x}{a} \times a = x$. Similarly if we have the fraction $\frac{3}{8}$ and multiply it by 8 we get the numerator 3 as a result.

109. We find the value of x from the equation $\frac{3x}{4} = 9$ by multiplying both sides by 4, and then dividing by the coefficient of x.

$$3x = 9 \times 4 = 36,$$
$$3x = 36$$
$$x = \frac{36}{3} = 12.$$

110. Consider the equation,

$$a + 2ax - c = 36 - 3ab.$$

First, we place a and c on the right side with the signs changed, that is we transpose the terms.

$$2ax = 36 - 3ab - a + c.$$

From Art. 11 it is evident that $2a$ is the coefficient of x; hence we divide both sides by $2a$.

$$x = \frac{36 - 3ab - a + c}{2a}.$$

111. Stated in words: The terms on the left-hand side which do not contain the unknown quantity should be placed on the right-hand side with the signs changed and all quantities in the same term with the unknown quantity should be transposed by multiplying or dividing both sides of the equation by them.

112. Find the value of P from the following formula:

$$P = \sqrt[3]{\frac{ABC}{1.4\,(B + \frac{A^2}{C^2})}}, \text{ when } A = 11,\ B = 1.71, \text{ and } C = 1.3.$$

Solution:

$$P = \sqrt[3]{\frac{11 \times 1.71 \times 1.3}{1.4\,(1.71 + \frac{121}{1.69})}}$$
$$= \sqrt[3]{\frac{24.453}{1.4 \times 73.308}}$$
$$= \sqrt[3]{.238} = .619 + \text{Ans.}$$

113. In solving equations remember the rule:

Terms containing unknown quantities may be transposed to the left-hand side of the equation by changing the signs. Terms con-

taining known quantities may be transposed to the right-hand side by changing the signs.

To find the value of the unknown quantity divide the right-hand side by the coefficient of the unknown quantity. It is readily seen that equations may be solved if the unknown quantity is on the right side. Thus $2 = x$ is the same as $x = 2$. It is better to leave the unknown quantity on the right side, if changing it necessitates many changes in the known quantities.

EXAMPLES FOR PRACTICE.

1. What is the value of x in the equation,

$\frac{x}{3} + 2\ (3 + 4) = 6 - 3 + 24$? Ans. $x = 39$

2. Find the value of x from the equation,

$6\,ax + 3\,a^2 = 3\,a - 3\,ab$ Ans. $x = \frac{1 - b - a}{2}$

3. What is the value of x in the equation,

$3\,a + 2\,b - x = 6$? Let $a = 2$ and $b = 1$ Ans. $x = 2$

4. What is the value of x in the following equation, where $a = 3$, $b = 2$, and $c = 1$?

$$2\,x - b = 20 + x - 3\,ac$$ Ans. $x = 13$

5. If $a = 5$, $b = 2$, and $c = 3$, what is the value of the equation,

$x + 4\,ax = 21 + 2\,c + \frac{9\,ab}{2\,c}$? Ans. $x = 2$

114. Suppose we have the equation,

$$H = \frac{PLAN}{33{,}000},$$

in which H = 50 horse-power, P = 49.12 pounds, L = $1\frac{1}{2}$ feet, and A = 113.1 square inches. The value of N is unknown but may be found by solving this equation.

We know that if $H = \frac{PLAN}{33{,}000}$, we can write it thus:

$$33{,}000\ H = PLAN \text{ or } PLAN = 33{,}000\ H.$$

This is the same as multiplying both sides by 33,000.

Now we may consider PLA the coefficient of N;

$$N = \frac{33000\ H}{PLA}$$

In other words we have placed the PLA on the other side.

By substituting the values given above we may solve for N.

$$N = \frac{33000\ H}{PLA}$$

$$= \frac{33000 \times 50}{49.12 \times \frac{3}{2} \times 113.1}$$

$$= 198 +$$

115. Suppose we wish to find the value of a in the following equation when $b = 2$, $c = 3$, $d = 4$, and $e = 5$.

$$6\,b\,c = a + \sqrt{d + e}$$

We can re-write the equation in order to get a upon the left side, thus,

$$a + \sqrt{d + e} = 6bc$$

$$a = 6bc - \sqrt{d + e}$$

$$a = 6 \times 2 \times 3 - \sqrt{4 + 5}$$

$$= 36 - \sqrt{9}$$

$$= 36 - 3$$

$$= 33$$

We place the $\sqrt{d + e}$ on the right-hand side of the equation with the sign changed. If the sign before the radical had been — instead of + we would have changed it to plus, thus,

$$a = 6\,b\,c + \sqrt{d + e}$$

When solving examples it often gives a more accurate answer to simply indicate operations instead of actually performing them.

116. Suppose we wish to obtain the value of the expression,

$$\sqrt{(\sqrt{890})^2 + 10.}$$

We might perform the work by first extracting the square root of 890, which is approximately 29.83. The above expression would then become,

$$\sqrt{(29.83)^2 + 10}$$

If now we continue the operations as indicated, we shall have,

$$\sqrt{889.8289 + 10}$$

$$\sqrt{899.8289}$$

$$29.99 +$$

117. Now suppose we take the same expression, $\sqrt{(\sqrt{890})^2 + 10}$ and simplify it in another way.

We will not extract the square root of the quantity 890 as before, because the *square of the square root* of 890, is 890. Hence, for the quantity $(\sqrt{890})^2$ we may write 890. The expression then becomes $\sqrt{890 + 10}$ or $\sqrt{900}$, the value of which is 30.

It is seen that by one method we get 29.99 + as the result and by the other method we get 30. The latter result is the exact one, and at the same time the operation is more simple.

118. Suppose we have the equation,

$$W = .0033 \times 10^{-7} \times n$$

We can write it thus,

$$W = .0033 \times \frac{1}{10^7} \times n$$

Now, any quantity having an exponent preceded by the minus sign, as a^{-2} can be written as a fraction. The quantity is placed in the denominator with the sign of the exponent changed; the numerator being 1. Thus,

$$a^{-2} = \frac{1}{a^2}; \quad x^{-6} = \frac{1}{x^6}; \quad 10^{-8} = \frac{1}{10^8} = \frac{1}{100,000,000}$$

We can write the above equation,

$$W = .0033 \times \frac{1}{10,000,000} \times n.$$

If the value of n is known the equation can be readily solved.

EXAMPLES FOR PRACTICE.

Solve the following equations:

1. $35x - 3(4 + 6) = 30x - 5$. Ans. $x = 5$.
2. $6x - (6 + 3x) = 12 + x - 14$. Ans. $x = 2$.
3. Given the equation $3a - 2bx = 14 + 3a - (4c - 22)$.
   $a = 4$, $b = 3$ and $c = 3$.
   Ans. $x = -4$.
4. Given the equation $6x - \frac{b}{a^{-4}} + c = 4ab + 4c$.
   $a = 2$, $b = 5$ and $c = 4$.
   Ans. $x = 22$.

MENSURATION.

119. Mensuration is the branch of mathematics which treats of the measurement of geometrical figures; that is, measurements of length, area and volume.

PRELIMINARY DEFINITIONS.

LINES.

120. A *line* has length only; it is produced by the motion of a point.

121. A *straight line* is one that has the same direction throughout. It is the shortest distance between two points.

122. A *curved line* is one that is constantly changing in direction; it is sometimes called a curve.

123. A *broken line* is one made up of several straight lines.

STRAIGHT LINE. CURVED LINE. BROKEN LINE. PARALLEL LINES.

124. *Parallel lines* are lines which lie in the same plane and are equally distant from each other at all points. They cannot meet, however far they may be produced (that is, extended in either direction.)

125. A *horizontal line* is one having the direction of a line drawn upon the surface of water that is at rest. It is a line parallel to the horizon or sea level.

126. A *vertical line* is one that has the direction of a thread suspended from its upper end and having a weight at its lower end. It is a line perpendicular to a horizontal line.

127. An *oblique line* is a line that is neither vertical nor horizontal.

128. In order to distinguish between lines, they are *lettered*. Thus if one end of a line is marked *a* and the other end *b* we call it "the line *a b*" or "the line *b a*."

SURFACES.

129. A surface has two dimensions; *length* and *breadth*

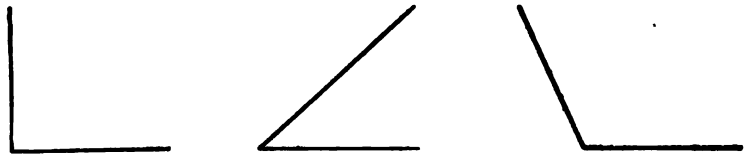
130. A *plane surface* is perfectly flat. A straight edge placed in any direction on a plane surface will be in contact at every point.

131. A *curved surface* is a surface no part of which is plane.

132. A *plane figure* is a combination of lines all of which lie in a plane surface.

ANGLES.

133. An angle is the difference in direction of two lines. The lines are called the *sides* and the point of meeting is called the *vertex*. The size of an angle depends upon the amount of divergence of the sides and is independent of the length of these lines.



RIGHT ANGLE. ACUTE ANGLE. OBTUSE ANGLE.

134. If one straight line meets another (at some point not an extremity) and the two angles thus formed are equal they are *right angles* and the lines are perpendicular to each other.

135. An *acute angle* is less than a right angle.

136. An *obtuse angle* is greater than a right angle.

137. Acute and obtuse angles are sometimes called *oblique angles*.

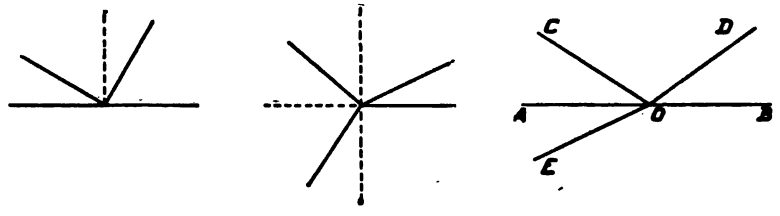
138. Angles are distinguished by placing letters on the lines forming the sides and at the vertex. Suppose the letter A is placed at the vertex and the letters B and C on the sides; we would read the angle as "the angle B A C" or "the angle C A B." The letter representing the vertex must be placed between those representing the sides. If but one letter is used it should be placed at the vertex and the angle is then designated as "the angle whose vertex is A" or simply "the angle A." In writing, the sign $\angle$ is often used in place of the word "angle."

139. As *two right angles* are formed when one line meets another (see Art. 134), we know that

The sum of all the angles about a point on one side of a straight line is equal to two right angles.

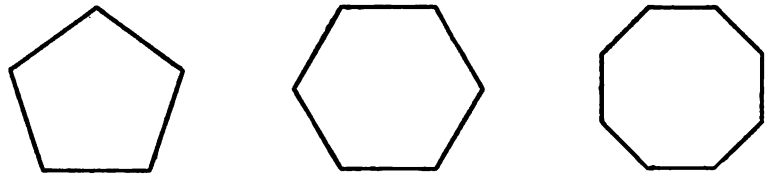
140. It also follows that the sum of all the angles about any point is equal to **four right angles.**

141. Consider the angles about the point O; angle A O C



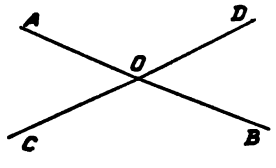
+ angle C O D + angle D O B is equal to two right angles. Angle A O E + angle E O B is equal to two right angles.

$\angle$ A O C + $\angle$ C O D + $\angle$ D O B + $\angle$ A O E + $\angle$ E O B = four right angles.



PENTAGON. HEXAGON. OCTAGON.

142. When two lines intersect they form four angles. The angles A O C and A O D are called **adjacent** angles because they have the same vertex O and the line A O is common, that is, it forms a side of each angle. The angles A O C and D O B are opposite angles.



From Art. 139 it follows that

$\angle$ A O C + $\angle$ A O D = two right angles.

It may also be proved that $\angle$ A O C = $\angle$ D O B.

POLYGONS.

143. A polygon is a plane figure bounded by straight lines.

144. The boundary lines are called the *sides* and the sum of the sides is called the *perimeter.*

145. Polygons are classified according to the number of sides.

A *triangle* is a polygon of *three* sides.

A *quadrilateral* is a polygon of *four* sides.

A *pentagon* is a polygon of *five* sides.

A *hexagon* is a polygon of *six* sides.

A *heptagon* is a polygon of *seven* sides.

An *octagon* is a polygon of *eight* sides.

A *decagon* is a polygon of *ten* sides.

A *dodecagon* is a polygon of *twelve* sides.

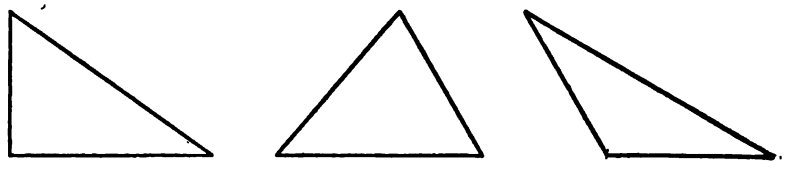
An *equilateral* polygon is one all of whose sides are equal.

An *equiangular* polygon is one all of whose angles are equal.

A *regular polygon* is one all of whose angles are equal and all of whose sides are equal.

TRIANGLES.

146. A *triangle* is a figure enclosed by three straight lines. The bounding lines are the *sides*, and the points of intersection of the sides are the *vertices*. The *angles* of a triangle are the angles formed by the sides.



RIGHT-ANGLED TRIANGLE. ACUTE-ANGLED TRIANGLE. OBTUSE-ANGLED TRIANGLE.

147. A *right-angled* triangle, often called a *right* triangle, is one that has a right angle. The longest side (the one opposite the right angle) is called the hypothenuse and the other lines are called sides or legs.

148. An *acute-angled* triangle is one that has all of its angles acute.

149. An *obtuse-angled* triangle is one that has an obtuse angle.

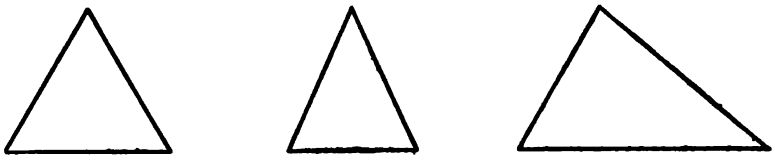
150. In an *equilateral* triangle all of the sides are equal.

151. If all of the *angles* of a triangle are equal, the figure is called an *equiangular* triangle.

152. A triangle is called *scalene* when no two of its sides are equal.

153. In an *isosceles* triangle two of the sides are equal.

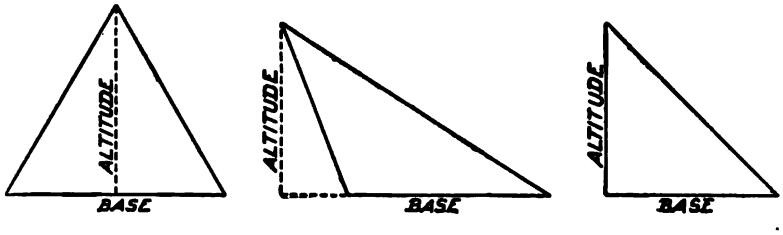
154. The *base* of a triangle is the lowest side; it is the side upon which the triangle is supposed to stand. Any side may, however, be taken as the base. In an isosceles triangle, the side



EQUILATERAL TRIANGLE. ISOSCELES TRIANGLE. SCALENE TRIANGLE.

which is not one of the equal sides is usually considered as the base.

155. The *altitude* of a triangle is the perpendicular drawn from the vertex to the base. In some triangles it is necessary to



produce the base so that the altitude may meet it. In a right triangle one leg may be considered as the base and the other the altitude.

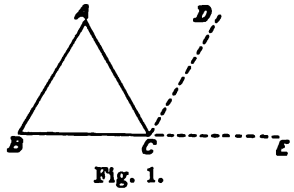


Fig. 1.

156. Let us consider Fig. 1. First produce (extend) B C to any point such as E and draw C D parallel to B A. It may be proved that angle A B C is equal to angle D C E and B A C equal to A C D. The angle A C B is of course equal to itself. Adding we have,

$$ABC + BAC + ACB = DCE + ACD + ACB$$

But since the sum of the angles about a point on one side of

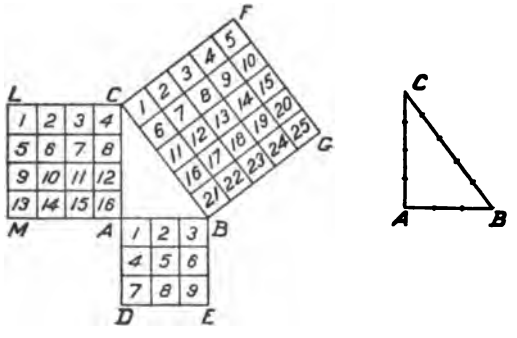
a straight line is equal to two right angles (see Art. 139) we have,

$$\text{D C E} + \text{A C D} + \text{A C B} = \text{two right angles;}$$
$$\text{therefore A B C} + \text{B A C} + \text{A C B} = \text{two right angles.}$$

157. Hence we may say that *the sum of all the angles of any triangle is equal to two right angles.*

158. The Right Triangle. If in the right triangle A B C the side A B is 3 inches long, and the side A C is four inches long the side B C (the hypothenuse) is 5 inches long.

By considering the accompanying figure we find that the square A B E D constructed on A B contains 9 squares. The



square L C A M constructed on A C contains 16 squares and the square C F G B constructed on B C contains 25 squares. Now,

$$9 + 16 = 25$$
$$3^2 + 4^2 = 5^2$$
$$\overline{\text{A B}}^2 + \overline{\text{A C}}^2 = \overline{\text{B C}}^2$$

159. Then we may say: *The square of the hypothenuse is equal to the sum of the squares of the other two sides.* This statement is true for **all right-angled triangles**; it does not, however, apply to triangles other than right.

This principle is very useful for finding the lengths of various lines.

Example. A right triangle has sides 6 inches and 10 inches long respectively; what is the length of the hypothenuse?

$$6^2 + 10^2 = \text{square of hypothenuse.}$$
$$36 + 100 = 136$$
$$\sqrt{136} = 11.66 + \text{inches.} \quad \text{Ans.}$$

160. Suppose we know the length of the hypothenuse and of one short side. We find the length of the other short side by means of the above.

Example. The hypothenuse of a right triangle is 24 feet long and one of the short sides is 9 feet. How long is the other short side?

$$24^2 - 9^2 = \text{the square of the required side.}$$
$$576 - 81 = 495$$
$$\sqrt{495} = 22.248 + \text{feet. Ans.}$$

Example. One side of a right triangle is 15 inches; if the hypothenuse is 21 inches, what is the other side?

$$21^2 - 15^2 = \text{the square of the other side.}$$
$$441 - 225 = 216$$
$$\sqrt{216} = 14.696 + \text{inches. Ans.}$$

EXAMPLES FOR PRACTICE.

1. One side of a right triangle is 8 feet and the other is 9 feet; what is the length of the hypothenuse?

Ans. 12 feet (about).

2. The hypothenuse of a right triangle is 18 inches and one side is 8 inches; what is the other side?

Ans. 17.748 + inches.

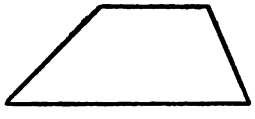
3. The two sides of a right triangle are 16 and 21 feet respectively. Find the length of the hypothenuse.

Ans. 26.4 feet.

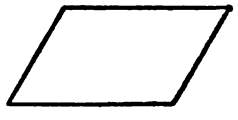
4. The hypothenuse of a right triangle is 23. What is the length of the other two sides if they are equal?

Ans. 16.263.

QUADRILATERAL.



TRAPEZOID.



PARALLELOGRAM.

QUADRILATERALS.

161. A *quadrilateral* is a plane figure bounded by four straight lines.

The *diagonal* of a quadrilateral is a straight line joining two opposite vertices.

162. A *trapezium* is a quadrilateral, no two of whose sides are parallel.

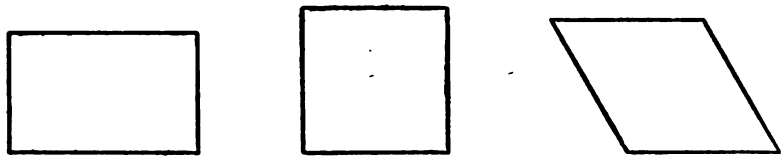
163. A *trapezoid* is a quadrilateral having two sides parallel.

The parallel sides are called the *bases* and the perpendicular distance between the bases is called the *altitude*.

164. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

The *altitude* of a parallelogram is the perpendicular distance between the *bases*, which are the parallel sides.

A diagonal of a parallelogram divides the parallelogram into *two equal triangles*.



RECTANGLE. SQUARE. RHOMBUS.

There are four kinds of parallelograms.

165. A *rectangle* is a parallelogram whose angles are right angles.

166. A *square* is a rectangle, all of whose sides are equal.

167. A *rhombus* is a parallelogram which has four equal sides; the angles, however, are not right angles.

168. A *rhomboid* is a parallelogram whose adjacent sides are unequal; the angles are not right angles.

CIRCLES.

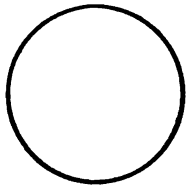
169. A *circle* is a plane figure bounded by a curved line, every point of which is equally distant from a point within, called the center.

170. The curve which bounds the circle is called the *circumference*. Any portion of the circumference is called the *arc*.

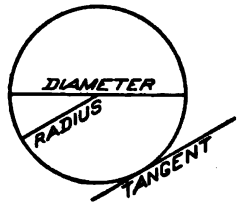
171. The *diameter* of a circle is a straight line drawn through the center and terminating in the circumference. A *radius* is a straight line joining the center with the circumference;

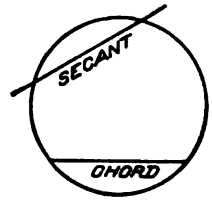
it has a length equal to one-half the diameter. All radii (plural of radius) are equal since every point of the circumference is equally distant from the center. All diameters are equal because a diameter is equal to two radii.

172. From the above it follows that two circles are equal when they have equal radii; if one circle is placed on the other so that the centers are in the same position, the circumferences will coincide.



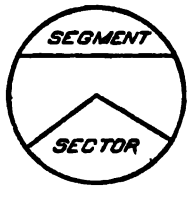
CIRCLE.

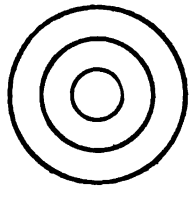




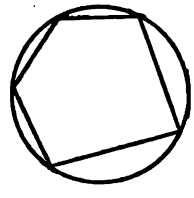
173. An arc equal to one-half the circumference is called a *semi-circumference.* A *quadrant* is one-quarter of a circumference. A quadrant may mean the sector, arc or angle.

174. A *chord* is a straight line joining the extremities of an arc. It is a line drawn across a circle that does not pass through the center.





CONCENTRIC CIRCLES.



INSCRIBED POLYGON.

175. A *secant* is a straight line which intersects the circumference in two points.

176. A *tangent* is a straight line which touches the circumference at only one point. It does not intersect the circumference. The point at which the tangent touches the circumference is called the *point of tangency* or *point of contact.*

It may be proved that a tangent is perpendicular to the extremity of a radius.

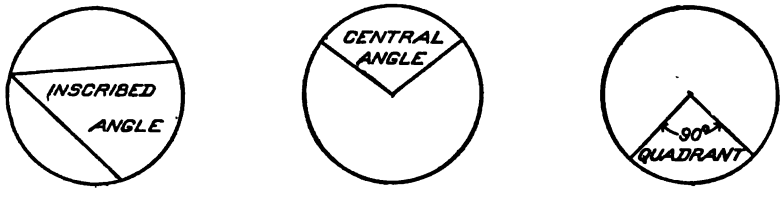
177. A *sector* of a circle is the portion or area included

between an arc and two radii drawn to the extremities of the arc.

178. A *segment* of a circle is the area included between an arc and its chord.

179. Circles are *tangent* when the circumferences touch at only one point and are *concentric* when they have the same center.

180. An *inscribed polygon* is one whose vertices lie in the circumference and whose sides are chords.



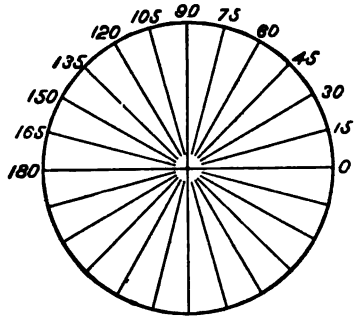
181. An *inscribed angle* is an angle whose vertex lies in the circumference and whose sides are chords. It is measured by one-half the intercepted arc.

182. A *central angle* is an angle whose vertex is at the center of the circle and whose sides are radii.

MEASUREMENTS.

ANGLES.

183. To measure an angle, describe an arc of any convenient radius having the center at the vertex of the angle. The arc included between the sides is the measure of the angle. With a given radius, the greater the divergence, the longer the arc. If several arcs are drawn having different radii, the intercepted arcs will have *different lengths* but they will all be the *same fraction* of the entire circumference.



184. To state the size of an angle as a fraction of the circumference is not always convenient, hence the entire circumference is divided into 360 equal parts called **degrees.** Angles are then measured in degrees. If an angle is measured by $\frac{1}{6}$ of

the circumference it contains 60 degrees, because $\frac{360}{6} = 60$. An angle of 45 degrees is measured by $\frac{1}{8}$ of the circumference.

185. For accurate measurement each degree is divided into 60 equal parts called **minutes** and each minute into 60 equal parts called **seconds.**

The symbols of degrees, minutes and seconds are °, ′, ″, respectively. To express 23 degrees, 47 minutes and 9 seconds, we write 23° 47′ 9″.

186. According to Art. 139, if lines are perpendicular to each other, right angles are formed. A right angle is measured by $\frac{1}{4}$ the circumference, hence it is measured by an arc of 90° since $\frac{360}{4} = 90$.

187. Two angles are **complementary** when their sum is equal to one right angle or 90 degrees; they are **supplementary** when their sum is 180 degrees. Thus, an angle of 33 degrees is the complement of one of 57 degrees because $33 + 57 = 90$ or $90 - 57 = 33$.

Angles of 148 degrees and 32 degrees are supplementary because $148 + 32 = 180$.

EXAMPLES FOR PRACTICE.

1. How many seconds in 180 degrees? Ans. 648,000.
2. How many right angles in 202° 30′? Ans. $2\frac{1}{2}$.
3. What is the complement of 37°?
4. What is the complement of 11° 47′ 3″?

Ans. 78° 12′ 57″.

5. What is the supplement of 131° 4′ 27″?

Ans. 48° 55′ 33″.

6. One angle of a scalene triangle is 17°; another angle is 21°. What number of degrees is the third?

Ans. 142°.

7. The vertical angle between the equal sides of an isosceles triangle is 25 degrees. How many degrees in each of the base angles?

Ans. 77° 30′.

AREAS OF POLYGONS.

RECTANGLES.

188. The amount of space covered by a plane figure is called

its **area.** In order to express area, we state how many times greater it is than another area taken as a unit.

189. The *unit* most commonly used is the **square foot,** or an area equal to a square measuring one foot on a side. For small areas the **square inch** may be used. If we say that a certain lot of land contains 10,000 square feet, we mean that its area is 10,000 times as great as an area of one square foot.

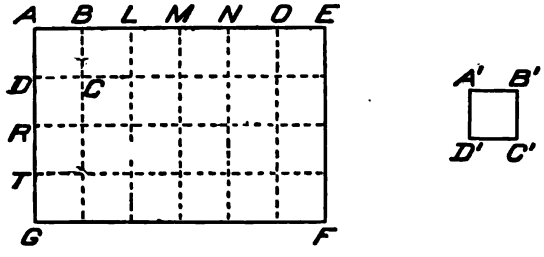


Fig. 2.

190. In Fig. 2 let A′ B′ equal 1 inch and A′ D′ equal 1 inch; then the area A′ B′ C′ D′ equals 1 square inch. The divisions A B, B L, L M, M N, N O and O E are each equal to A′ B′. Similarly the divisions A D, D R, R T and T G are equal to A′ D′.

It will be seen that the rectangle A E F G contains 24 of these small squares, each equal to the unit area A′ B′ C′ D′. Since there are 6 divisions in the base and 4 in the altitude, and $6 \times 4 = 24$, we have the following rule:

191. To find the area of a rectangle, multiply the base by the altitude.

Example. The base of a rectangle is 18 feet and the altitude is 7 feet; what is the area?

$$base \times altitude = area$$
$$18 \times 7 = 126$$

Ans. 126 square feet.

Another Example. The *area* of a rectangle is 39 square inches and the base is 13 inches; what is the altitude?

$$base \times altitude = area.$$

or
$$altitude = \frac{area}{base}$$
$$= \tfrac{3}{1}\tfrac{9}{3} = 3$$

Ans. 3 inches.

192. Since the *square* is a rectangle, its area is found by the same rule. In the square, however, the base equals the altitude; hence the area is equal to the square of one side.

PARALLELOGRAMS.

193. In the parallelogram shown in Fig. 8 we know that the base is D C, and the altitude A G, or any line perpendicular to D C. If we produce D C to F to meet the line perpendicular to the bases from E, the right triangle E F C is formed. This triangle may be proved equal to the right triangle A G D. If we take the area A G D from the parallelogram and add the equal area E F C, we do not change the area of the parallelogram. Hence,

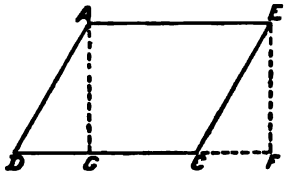


Fig. 8.

$$\text{area A E F G} = \text{area A E C D.}$$

We know that the area of the rectangle A E F G is equal to the product of its base and altitude. From this we can state:

194. The area of a parallelogram is equal to the product of the base and altitude.

Example. The altitude of a parallelogram is 9 inches and the base is 12 inches. What is the area?

$$\text{area} = \text{base} \times \text{altitude}$$
$$\text{area} = 12 \times 9$$
$$\text{area} = 108 \text{ square inches.}$$

195. $$\text{Base} = \frac{\text{Area}}{\text{Altitude}} \qquad \text{Altitude} = \frac{\text{Area}}{\text{Base}}$$

TRIANGLES.

196. If we draw the diagonal A F (Fig. 4), the parallelogram A C F B is divided into two equal triangles — B A F and A F C. The area of the parallelogram is equal to the base multiplied by the altitude. Hence the area of the triangle is equal to $\frac{1}{2}$ the base multiplied by the altitude. This may be expressed

$$\text{area} = \frac{\text{base} \times \text{altitude}}{2}$$
$$= \frac{\text{base}}{2} \times \text{altitude}$$
$$= \text{base} \times \frac{\text{altitude}}{2}$$

197. The area of a triangle equals one-half the product of the base and the altitude.

Example. The base of a triangle is 6 and the altitude is 3. What is the area?

$$\text{area} = \frac{\text{base} \times \text{altitude}}{2}$$

$$= \frac{6 \times 3}{2} = 9.$$

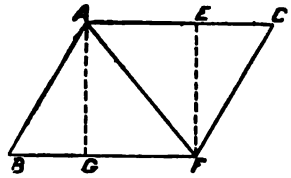


Fig. 4.

198. In case the **three sides are known** the area is found as follows:

(1) Add together the three sides and divide the sum by 2.

(2) Subtract from the half sum (1) the three sides severally.

(3) Multiply the half sum (1) and the three remainders (2), and extract the square root of the product.

Example. The three sides of a triangle are 24, 18 and 16 inches respectively. What is the area?

The half sum (1) is $\frac{24 + 18 + 16}{2} = 29$

The three remainders (2) are $\begin{cases} 29 - 24 = 5 \\ 29 - 18 = 11 \\ 29 - 16 = 13 \end{cases}$

$$\text{The area} = \sqrt{29 \times 5 \times 11 \times 13}$$
$$= \sqrt{20{,}735}$$
$$= 144 \text{ — square inches.}$$

199. The Right Triangle. The area of the right triangle is equal to one-half the product of the base and altitude. If both are given, the problem is very simple; if the base and hypothenuse are given, the altitude may be found by the principle explained in Arts. 158 and 159.

$$\text{area} = \frac{AB \times AC}{2}. \quad \text{(See Fig. 5.)}$$

200. The Isosceles Triangle. The isosceles triangle A B C is equal to the sum of the two right triangles A B D and B C D.

If the base and altitude are known, the principle explained in Art. 197 may be used.

If the base A C and the side A B are known, the principle explained in Art. 159 may be followed to find the altitude, and then the area is found according to Art. 197. As the side B C = side A B, the area may be found according to Art. 198, as the three sides are given.

201. The Equilateral Triangle. The equilateral triangle is an isosceles triangle; hence any of the above rules may be used according to the data.

Suppose the side B C = 1. The side A C also equals 1, and D C = $\frac{1}{2}$.

$$\text{From Art. 159, } \overline{BD}^2 = \overline{BC}^2 - \overline{DC}^2$$
$$= 1^2 - (\tfrac{1}{2})^2$$
$$= 1 - \tfrac{1}{4} = \tfrac{4}{4} - \tfrac{1}{4}$$
$$= \tfrac{3}{4}$$
$$BD = \sqrt{\tfrac{3}{4}} = \sqrt{.75}$$
$$= .866$$

202. To find the altitude multiply the side by .866.

$$\text{Area of triangle A B C} = \frac{BD}{2} \times AC$$
$$= (.433 \times AC) \times AC$$
$$= .433 \times \overline{AC}^2$$

203. The area of an equilateral triangle is equal to the square of the side multiplied by .433.

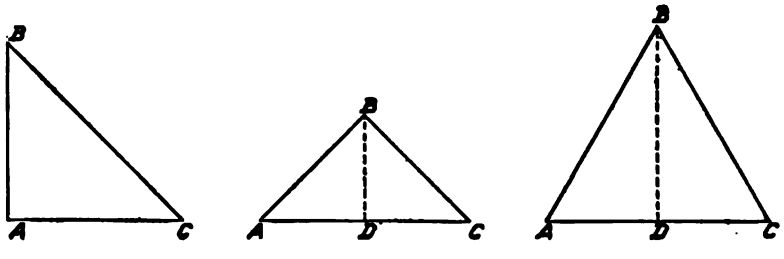


Fig. 5.

The above rule will be found very useful, on account of its simplicity, and because the results are sufficiently accurate for all practical purposes.

Example. The side of an equilateral triangle is 6 inches. What is the area?

$$\begin{aligned}\text{Area} &= (\text{side})^2 \times .433\\ &= 6^2 \times .433\\ &= 36 \times .433\\ &= 15.588 \text{ square inches.}\end{aligned}$$

Another example. The side of an equilateral triangle is 8 inches. What is the area?

$$BC = 8 \qquad DC = 4$$

$$\begin{aligned}BD &= \sqrt{64 - 16}\\ &= \sqrt{48}\\ &= 6.928\\ \text{area} &= \frac{BD \times AC}{2}\\ &= \frac{6.928 \times 8}{2}\\ &= 27.71 \text{ square inches.}\end{aligned}$$

204. $\text{Base} = \frac{2 \times \text{Area}}{\text{Altitude}}$ $\qquad \text{Altitude} = \frac{2 \times \text{Area}}{\text{Base}}$.

EXAMPLES FOR PRACTICE.

1. The base of a right-angled triangle is 12 feet and the hypothenuse is 20 feet. What is the length of the perpendicular?

Ans. 16 feet.

2. The base of a parallelogram is 16.25 feet and the altitude is 7.5 feet. What is the area? Ans. 121.875 square feet.

3. The foot of a ladder is 15 feet from the base of a building, and the top reaches a window 36 feet above the base. What is the length of the ladder? Ans. 39 feet.

4. An equilateral triangle has an area of 21.217 square inches. What is the length of the side? Ans. 7 inches.

5. An isosceles triangle has a base 16 inches long and the equal sides 18 inches long. What is the area?

Ans. 128.96 square inches.

6. The base of a triangle is 20 feet and the altitude is 18 feet. What is the side of a square having the same area?

Ans. 13.41 feet.

7. What is the area of a triangle having sides 16, 12 and 10 feet? Ans. 59.9 square feet.

8. What is the area of an equilateral triangle if the side is 11 feet? Ans. 52.393 square feet.

TRAPEZOID.

205. If in the trapezoid shown in Fig. 6, a diagonal D B is drawn, the trapezoid is divided into two triangles. The altitude of both triangles is the same; the base of triangle D B C is the lower base and the base of D A B is the upper base. To find the area we proceed as follows:

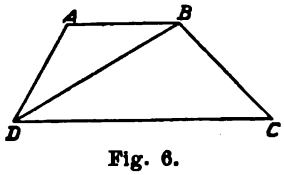


Fig. 6.

$$\text{Area D B C} = \text{altitude} \times \frac{1}{2}\,\text{D C}.$$

$$\text{Area D A B} = \text{altitude} \times \frac{1}{2}\,\text{A B}.$$

$$\text{Total area} = \text{altitude} \times \frac{\text{D C} + \text{A B}}{2}.$$

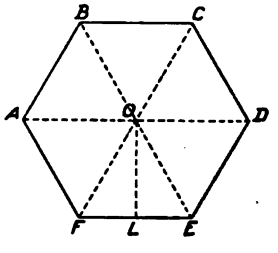
206. The area of a trapezoid equals the product of the altitude and one-half the sum of the bases.

Example. If the bases of a trapezoid are 10 inches and 8 inches respectively and the altitude is 7 inches, what is the area?

$$\text{Area} = 7\left(\frac{10+8}{2}\right) = 63 \text{ square inches.}$$

HEXAGON.

207. A regular polygon is one all of whose sides and angles are equal. We may divide a regular hexagon into six equal equilateral triangles. The area of the triangle F O E may be found by any of the rules given for equilateral triangles.



$$\text{O L} = \sqrt{\overline{\text{O E}}^2 - \overline{\text{L E}}^2}$$

$$\text{Area} = \frac{\text{O L} \times \text{F E}}{2}$$

$$\text{Area of hexagon} = \frac{\text{O L} \times \text{F E}}{2} \times 6 = \text{O L} \times \text{F E} \times 3.$$

We know that the area of an equilateral triangle may be found by multiplying the square of the side by .433.

$$\text{Area} = (\text{side})^2 \times .433.$$
$$\text{Area of hexagon} = (\text{side})^2 \times .433 \times 6.$$
$$= (\text{side})^2 \times 2.598.$$

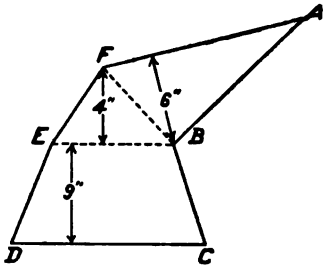
Example. The side of regular hexagon is 5 feet long. What is the area?

$$5^2 \times 2.598 = 64.95 \text{ square feet.}$$

POLYGONS.

208. To find the area of a polygon we divide it into triangles and take the sum of the areas.

209. Suppose we wish to know the area of the given polygon A B C D E F. First we divide it into the triangles A B F and E F B and the trapezoid B C D E. This is done by drawing the lines F B and E B. We find the altitudes to be 6 inches, 4 inches and 9 inches respectively. If the line A F is 14 inches, D C 15 inches and E B 11 inches, the areas are as follows:



$$\text{Area A B F} = \frac{6 \times 14}{2} = 42 \text{ square inches.}$$

$$\text{Area B E F} = \frac{4 \times 11}{2} = 22 \text{ square inches.}$$

$$\text{Area B C D E} = \frac{9 \times (11 + 15)}{2} = 117 \text{ square inches.}$$

The total area is 42 + 22 + 117, or 181 square inches.

EXAMPLES FOR PRACTICE.

1. The bases of a trapezoid are 23 feet and 11 feet; the altitude is 9 feet. What is the area?

Ans. 158 square feet.

2. What is the area of a regular hexagon having a perimeter of 42 feet?

Ans. 127.3 square feet.

3. If the area of a regular hexagon is 314.358 square inches, what is the length of the altitude of one of the equal equilateral triangles?

Ans. 9.526 inches.

4. *Find the area of a trapezium, a diagonal of which measures 42 feet, and the perpendicular from one vertex to the diagonal is 16 feet, and from the other vertex 18 feet.

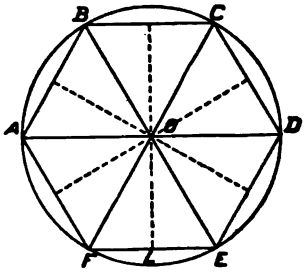
Ans. 714 square feet.

5. What is the area of the trapezium a diagonal of which is 35 feet 6 inches long, the perpendiculars to the diagonal from the other two vertices being 9 and 3 feet respectively?

Ans. 213 square feet.

CIRCLE.

210. Suppose we have a regular hexagon A B C D E F inscribed in the circle whose center is O. The hexagon may be divided into six triangles by drawing the radii connecting the vertices A, B, C, D, etc., with the center. These triangles are all equal and the area of each is equal to one-half the product of the base and altitude. We may say that it is equal to one-half the altitude multiplied by the base. The area of six triangles is equal to one-half the altitude O L of one of them multiplied by six times the base or the perimeter.



$$\text{Area} = \frac{\text{altitude}}{2} \times \text{perimeter.}$$

211. Now if the number of sides of the polygon is increased, the altitude will increase and the perimeter will be more nearly equal to the circumference of the circle. If the number of sides is increased indefinitely, the altitude of each triangle will be practically the same as the radius of the circle and the perimeter will be very nearly the same as the circumference. The area of the circle will be that of the inscribed polygon of an infinite number of sides, or

$$\text{Area of circle} = \frac{\text{radius}}{2} \times \text{circumference.}$$

It has been found that the diameter of a circle multi-

*NOTE. The diagonal connects two vertices. The perpendiculars are from the other two vertices to the diagonal. They are the altitudes of the two triangles of which the diagonal forms the base.

plied by the constant 3.1416 is equal to the circumference. The constant 3.1416 expresses the ratio of the circumference to the diameter, and is called Pi; it is usually designated by the Greek letter π.

212. The circumference of a circle *is equal to the diameter multiplied by* 3.1416.

From the above we know that

$$\begin{aligned}\text{Area of circle} &= \frac{\text{radius}}{2} \times \text{circumference.}\\ &= \frac{\text{radius}}{2} \times \text{diameter} \times 3.1416.\\ &= \frac{\text{radius}}{2} \times 2 \times \text{radius} \times 3.1416.\\ &= \text{radius} \times \text{radius} \times 3.1416.\end{aligned}$$

213. The area of a circle *equals the square of the radius multiplied by* 3.1416.

Example. What is the area of a circle whose radius is 4 feet?

$$\begin{aligned}\text{Area} &= (\text{radius})^2 \times 3.1416\\ &= 4 \times 4 \times 3.1416\\ &= 50.2656 \text{ square feet.}\end{aligned}$$

Another example. If the diameter of a circle is 12 inches, what is the area?

$$\begin{aligned}\text{Radius} &= \frac{\text{diameter}}{2}\\ &= \frac{12}{2} = 6 \text{ inches.}\\ \text{Area} &= 6 \times 6 \times 3.1416\\ &= 113.0976 \text{ square inches.}\end{aligned}$$

The above example may also be solved as follows:

$$\begin{aligned}\text{Area} &= \text{radius} \times \text{radius} \times 3.1416\\ &= \frac{\text{diameter}}{2} \times \frac{\text{diameter}}{2} \times 3.1416\\ &= \frac{\text{diameter} \times \text{diameter}}{4} \times 3.1416\\ &= \text{diameter} \times \text{diameter} \times \frac{3.1416}{4}\end{aligned}$$

Since $\frac{3.1416}{4} = .7854.$

$$\begin{aligned}\text{Area} &= 12 \times 12 \times .7854\\ &= 113.0976 \text{ square inches.}\end{aligned}$$

214. The area of a circle *is equal to the square of the diameter multiplied by* .7854.

215. By considering Art. 211 to Art. 214 and using the principles of Elementary Algebra we have the following:

$$\text{Circumference} = 3.1416 \times \textit{diameter}$$
$$= 2 \times 3.1416 \times \textit{radius}$$
$$= \frac{4 \times \textit{area}}{\textit{diameter}}$$

$$\text{Area} = \textit{radius} \times \textit{radius} \times 3.1416$$
$$= \textit{diameter} \times \textit{diameter} \times .7854$$
$$= \textit{circumference} \times \frac{\textit{radius}}{2}$$

$$\text{Diameter} = \frac{\textit{circumference}}{3.1416}$$
$$= \sqrt{\frac{4 \times \textit{area}}{3.1416}}$$

$$\text{Radius} = \frac{\textit{circumference}}{2 \times 3.1416}$$
$$= \sqrt{\frac{\textit{area}}{3.1416}}$$

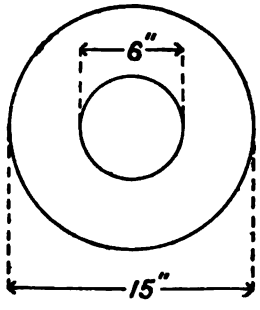


Fig. 7.

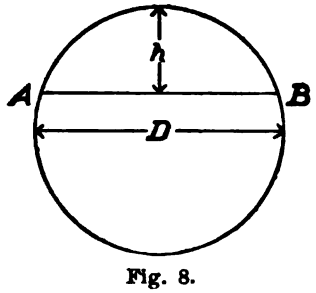


Fig. 8.

216. Circular Ring. To obtain the area of a flat circular ring, first find the area of the hole and subtract it from the area enclosed by the larger circumference.

In Fig. 7 we have:

Area of large circle $= .7854 \times \overline{15}^2 = 176.71$ square inches
Area of small circle $= .7854 \times 6^2 = 28.27$ square inches
Area of ring $= 148.44$ square inches

217. Sector. In finding the area of a sector we determine

the fraction of the area of a circle. Thus if the sector has an arc of 60°, the area of the sector would be 60 ÷ 360 = one-sixth of the area of the circle of which the sector forms a part. Similarly a sector having an arc of 45° has one-eighth the area of a circle having a radius equal to one of the sides of the sector.

218. Segment. There are many rules and tables for finding the area of a segment. The following formula is very simple and gives approximate results. The derivation of this formula is very difficult, as it involves higher mathematics; the formula may be used, however, by students and engineers who wish a simple formula giving approximate results.

$$\text{Area of segment A} = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}$$

In this formula h = height of segment and D = the diameter of the circle. See Fig. 8.

219. Suppose we wish to find the area of a segment when the diameter of the circle is 12 feet and the height of the segment is 4 feet.

In this case $h = 4$ feet and D = 12 feet. Substituting these values in the formula we have,

$$A = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}$$

$$= \frac{4 \times 4^2}{3}\sqrt{\frac{12}{4} - .608}$$

$$= \frac{64}{3}\sqrt{3 - .608}$$

$$= \frac{64}{3}\sqrt{2.392} = \frac{64}{3} \times 1.55$$

$$= 33.06 + \text{ square feet.}$$

EXAMPLES FOR PRACTICE.

1. The diameter of a circle is 9 feet; what is the area?

Ans. 63.617 square feet.

2. What is the circumference of a circle 17 inches in diameter?

Ans. 53.407+ inches.

3. If a circle is 4 feet in diameter, what is the area of a sector having an arc of 80°? Ans. 2.7925 square feet.

4. The area of a certain circle is 283.53 square feet; what is the circumference?

Ans. 59.69 feet.

5. The area of a circle is 99.402 square inches; what is the circumference?

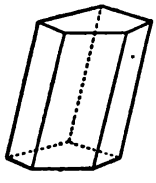
Ans. 35.343 inches.

SOLIDS.

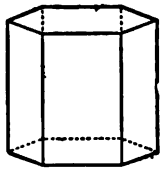
220. A solid has three dimensions — length, breadth and thickness.

PRISMS.

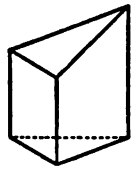
221. A prism is a solid having two opposite faces called *bases*, equal and parallel; the other faces, called lateral faces, are parallelograms.



PRISM.



RIGHT PRISM.



TRUNCATED PRISM.

222. The *altitude* of a prism is the perpendicular distance between the bases.

223. The combined area of all the lateral faces is called the *lateral area.*

The *total area* or entire surface is the lateral area plus the areas of the bases.

224. The intersections of the faces are called the *lateral edges.*

225. Prisms are *triangular*, *rectangular*, *hexagonal*, etc., according to the shape of the bases.

226. A *right* prism is one whose lateral edges are perpendicular to the bases.

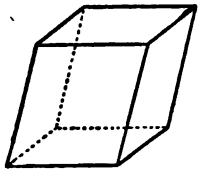
227. A *regular* prism is a right prism having regular polygons for bases.

228. A *parallelopiped* is a prism whose bases are parallelograms. If all the edges are perpendicular to the bases, it is called a right parallelopiped.

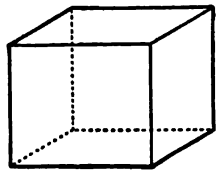
229. A *rectangular* parallelopiped is a right parallelopiped whose bases are rectangles; all the faces are rectangles.

230. A *cube* is a rectangular parallelopiped all of whose faces are squares.

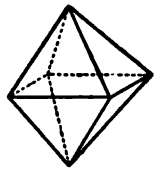
231. A *truncated* prism is the portion of a prism included between its base and a plane cutting all the lateral edges.



PARALLELOPIPED.



RECTANGULAR PARALLELOPIPED.

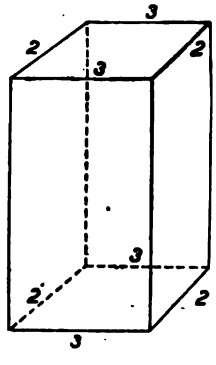


OCTAHEDRON.

232. Lateral Area. Suppose we have a box 3 feet long, 2 feet wide and 6 feet deep. Let us consider the top and bottom as the bases.

The area of the front face is evidently $6 \times 3 = 18$ square feet; the area of the back face is also 18 square feet, as it is equal to the front face.

The area of the left-hand face is $2 \times 6 = 12$ square feet, and the area of the right-hand face is the same.



The convex surface is equal to

$$2 \times 18 = 36$$
$$2 \times 12 = 24$$
$$\overline{60} \text{ square feet.}$$

233. It is readily seen that had we multiplied the perimeter of the base by the altitude we would have obtained the same result.

Perimeter of base $= 3 + 2 + 3 + 2 = 10$ feet.

Altitude $= 6$ feet.

Lateral Area $= 10 \times 6 = 60$ square feet.

234. ***The lateral area of any right prism is equal to the product of the perimeter of the base and altitude.***

This rule may be used for all right prisms, whether the base is a triangle, square, parallelogram or hexagon.

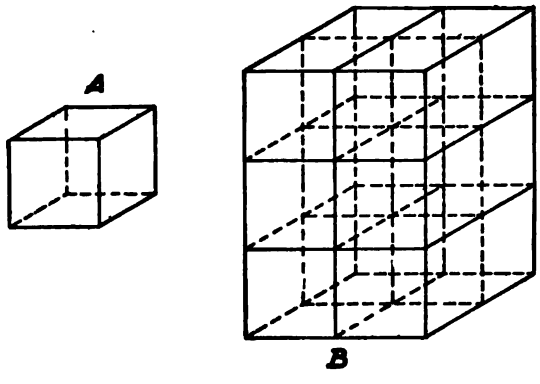
Example. What is the lateral area of a right prism whose altitude is 11 feet and each side of the hexagonal base is 4 feet?

The perimeter of the base $= 6 \times 4 = 24$ feet.
Altitude $= 11$ feet.
Lateral area $= 24 \times 11 = 264$ square feet.

235. Total Area. *To find the total area of a prism, add the areas of the bases to the lateral area.*

236. In the prism described in Art. 232, the lateral area was found to be 60 square feet. The area of the lower base is equal to $3 \times 2 = 6$ square feet and the area of the upper base is the same.

$$\text{Total area} = 60 + (2 \times 6) = 72 \text{ square feet.}$$



237. Volume. The unit of volume is a cube whose edges are each equal to the unit of length (inch, foot, etc.). The volume is expressed as being equal to the number of times the unit volume is contained in the prism.

238. By considering the above figures we see that the unit of volume A is contained 4 times in each layer of the prism B. As there are 3 layers, it is contained $4 \times 3 = 12$ times. If the unit of volume is the cubic foot, the prism B would contain 12 cubic feet. The area of the base is 4 square feet and the altitude is 3 feet. Hence we may say:

239. *The volume of any right prism is equal to the product of the area of the base and the altitude.*

Example. Find the volume of a block of stone 3 feet long, 2 feet wide and 7 feet high.

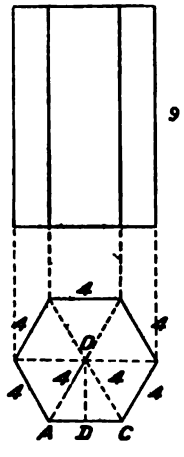
The area of the base $= 2 \times 3 = 6$ square feet.
The volume $= 6 \times 7 = 42$ cubic feet.

We may consider any other side as the base; the result is the same.

$2 \times 3 \times 7 = 42.$
$7 \times 3 \times 2 = 42.$
$7 \times 2 \times 3 = 42.$

240. Let us consider an *hexagonal prism* having an altitude of 9 feet and each side of the base 4 feet.

To find the volume we must first find the area of the base. This may be done according to Art. 207, or by finding the altitude O D of the triangle A O C, and then proceeding as in Art. 197. After finding the area of the base, multiply by the altitude.



Area of base $= 16 \times 2.598$
$= 41.568$ square feet
Volume $= 41.568 \times 9$
$= 374.112$ cubic feet.

241. The lateral and total areas are found as before. (See Arts. 232 and 235.)

Lateral Area $= 24 \times 9$
$= 216$ square feet
Total Area $= 216 + (2 \times 41.568)$
$= 299.136$ square feet.

EXAMPLES FOR PRACTICE.

1. Find the volume of a cube whose edges are each 6 feet 6 inches.

Ans. $274\frac{5}{8}$ cubic feet.

2. Find the volume of a triangular prism whose altitude is 20 feet and each side of the base 4 feet.

Ans. 138.56+ cubic feet.

3. Find the lateral area of a prism having an altitude of 7 feet and its base a pentagon, each side of which is 4 feet.

Ans. 140 square feet.

4. What is the entire surface of a parallelopiped 8 feet 9 inches long, 4 feet 8 inches wide and 3 feet 3 inches high?

Ans. $168\frac{7}{8}$ square feet.

5. Find the volume of an hexagonal prism 17 feet high, having a base each side of which is 7 feet.

Ans. 2,164.1 cubic feet.

6. What is the total area of a cube whose edges are each 7 inches?

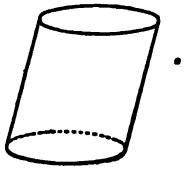
Ans. 294 square inches.

7. What is the weight of a cast iron block 6 inches long, 5 inches wide and 3 inches high? Cast iron weighs about .26 pound per cubic inch.

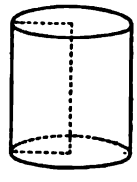
Ans. 23.4 pounds.

CYLINDERS.

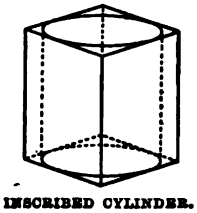
242. A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes intersecting this surface. It may be considered as being a prism whose base is a polygon of an infinite number of sides.



CYLINDER.



RIGHT CYLINDER.



INSCRIBED CYLINDER.

243. The parallel faces are called the *bases.*

244. The *altitude* is the perpendicular distance between the bases.

245. A *circular cylinder* is a cylinder whose base is a circle.

246. A *right circular cylinder*, or cylinder of revolution, is a cylinder generated by the revolution of a rectangle about one side as an axis.

247. A cylinder may be inscribed in or circumscribed about a prism.

248. The cylindrical area is called the *lateral area.* The *total area* is the sum of the lateral area and the areas of the bases.

249. Lateral Area. *The lateral area of a cylinder is equal to the circumference of the base multiplied by the altitude.*

250. Suppose we have a cylinder 15 inches high and 6 inches in diameter.

The circumference $= 3.1416 \times 6 = 18.8496$ inches.
The lateral area $= 18.8496 \times 15 = 282.744$ square inches.

251. Total Area. *To find the total area of a cylinder, add the areas of the bases to the lateral area.*

252. In the above cylinder the area of the base is,

$36 \times .7854 = 28.2744$ square inches.
The total area $= 282.744 + 2\ (28.274)$
$= 339.292$ square inches.

253. Volume. *The volume of a cylinder is equal to the area of the base multiplied by the altitude.*

Example. A cylinder is 6 feet in height and 4 feet in diameter; what is the volume?

Area of base $= 4 \times 4 \times .7854$
$= 12.5664$ square feet.
Volume $= 12.5664 \times 6$
$= 75.3984$ cubic feet.

EXAMPLES FOR PRACTICE.

1. What is the volume of a cylinder whose altitude is $8\frac{1}{2}$ feet and base 3 feet in diameter?

Ans. 60.08+ cubic feet.

2. Find the total area of a cylindrical boiler 56 inches in diameter and 16 feet long.

Ans. 268.78 square feet.

3. How many gallons of water will the above boiler hold if it contains no tubes?

Ans. 2,047 gallons.

PYRAMID.

254. A pyramid is a solid whose base is a polygon and whose sides are triangles.

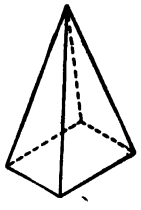
255. The vertices of the triangles meet to form the *vertex* of the pyramid.

256. The *altitude* of the pyramid is the perpendicular distance from the vertex to the base.

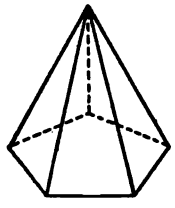
257. A pyramid is triangular, quadrangular, etc., according to the shape of the base.

258. A *regular* pyramid is one whose base is a regular polygon and whose vertex lies in a perpendicular erected at the center of the base.

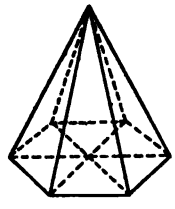
259. The *slant height* of a regular pyramid is a line drawn from the vertex perpendicular to a side of the base. It is the altitude of one of the triangles which form the sides.



PYRAMID.



REGULAR PYRAMID.



HEXAGONAL PYRAMID.

260. The lateral edges of a pyramid are the intersections of the faces.

261. Lateral Area. The lateral area is the combined area of all the triangles forming the sides.

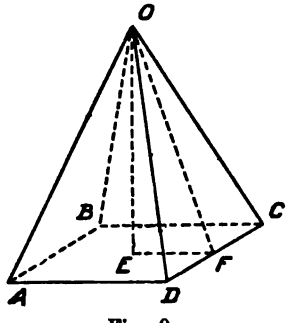


Fig. 9.

262. As the area of each triangle is equal to the product of the base and one-half the altitude, *the lateral area of the pyramid is equal to the perimeter multiplied by one-half the slant height.*

Example. Suppose a pyramid having a slant height of 14 feet has a regular pentagon for a base, each side of which is 5 feet. What is the lateral area?

Perimeter of base $= 5 \times 5 = 25$ feet.

One-half the slant height $= 7$ feet.

Lateral area $= 25 \times 7 = 175$ square feet.

263. If the slant height is not given it can usually be found by means of the principle explained in Art. 158.

264. The pyramid shown in Fig. 9 has an altitude O E of 12 feet and each side of the square base is 8 feet. We know that

E F = 4 feet. Since the angle O E F is a right angle (Art. 159),

$$\overline{O F}^2 = \overline{O E}^2 + \overline{E F}^2$$
$$= 144 + 16$$
$$= 160$$
$$O F = \sqrt{160} = 12.65 \text{ (nearly)}$$

$$\text{The lateral area} = \frac{12.65 \times 32}{2}$$
$$= 202.4 \text{ square feet.}$$

265. The Total Area *is equal to the lateral area plus the area of the base.*

In the above example the area of the base is 64 square feet. (Art. 192.) The total area is,

$64 + 202.4 = 266.4$ square feet.

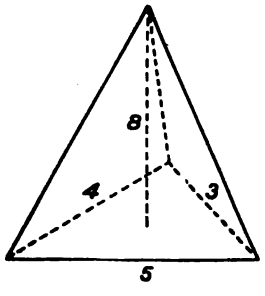


Fig. 10.

266. Volume. *The volume of a pyramid is equal to the area of the base multiplied by one-third the altitude.*

267. Example. Suppose the triangular pyramid shown in Fig. 10 has the following dimensions: altitude 8 feet, sides of base 5, 4 and 3 feet. What is the volume?

Area of base:

$$\text{The half sum (1)} = 12 \div 2 = 6$$

$$\text{The three remainders (2) are} \begin{cases} 6 - 5 = 1 \\ 6 - 4 = 2 \\ 6 - 3 = 3 \end{cases}$$

$$\text{The area} = \sqrt{6 \times 1 \times 2 \times 3}$$
$$= \sqrt{36}$$
$$= 6 \text{ square feet.}$$

$$\text{Volume} = \text{area of base} \times \tfrac{1}{3} \text{ altitude}$$
$$= 6 \times \tfrac{8}{3}$$
$$= 16 \text{ cubic feet.}$$

EXAMPLES FOR PRACTICE.

1. A triangular pyramid is 9 inches in height and each side of the base is 4 inches long. Find the volume.

Ans. 20.784 cubic inches.

2. Find the lateral area, total area and volume of a pyramid having a slant height of 72 inches and a square for a base, the side of which is 8 feet.

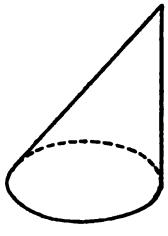
$$\text{Ans.} \begin{cases} \text{Lateral Area} = 96 \text{ square feet.} \\ \text{Total Area} = 160 \text{ square feet.} \\ \text{Volume} = \dfrac{64\sqrt{20}}{3} = 95.4 \text{ cubic feet.} \end{cases}$$

3. Find the lateral area and volume of a hexagonal pyramid each side of whose base is 6 feet and whose altitude is 12 feet.

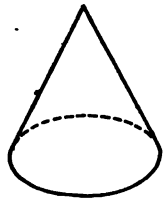
$$\text{Ans.} \begin{cases} \text{Lateral Area} = 235.26 \text{ square feet.} \\ \text{Volume} = 374.112 \text{ cubic feet.} \end{cases}$$

CONES.

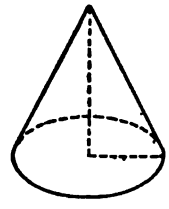
268. A cone is a solid bounded by a conical surface and a plane which cuts the conical surface. It may be considered as a pyramid of an infinite number of sides.



CONE.



CIRCULAR CONE.



CONE OF REVOLUTION.

269. The curved surface is called the *lateral area*, and the plane is called the *base*. The convex surface tapers to a point called the *vertex*.

270. The *altitude* of a cone is the perpendicular distance from the vertex to the base.

271. An *element* of a cone is any straight line from the vertex to the perimeter of the base.

272. A *circular cone* is a cone whose base is a circle.

273. A *right circular cone*, or *cone of revolution*, is a cone whose axis is perpendicular to the base. It may be generated by the revolution of a right triangle about one of the sides as an axis.

274. Lateral Area. The lateral area of a cone is found in the same way as in the case of a pyramid. Multiply the perimeter of the base by one-half the slant height.

Example. The base of a cone is 12 inches in diameter and the slant height is 16 inches. What is the lateral area?

$$\text{Perimeter of base} = 3.1416 \times 12$$
$$= 37.699 \text{ inches}$$
$$\text{Lateral area} = 37.699 \times \frac{16}{2}$$
$$= 301.592 \text{ square inches.}$$

275. To find the total area, add the area of the base.

In the above example, the area of the base is,

$$3.1416 \times 36 = 113.0976$$
$$\text{Total area} = 113.0976 + 301.592 = 414.6896 \text{ square inches.}$$

276. Volume. The volume of a cone is equal to the product of the area of the base and one-third the altitude.

Example. The altitude of a cone is 18 inches and the radius of the base is 2 inches. What is the volume?

$$\text{Area of base} = 3.1416 \times 4 = 12.5664$$
$$\text{Volume} = 12.5664 \times 6$$
$$= 75.3984 \text{ cubic inches.}$$

277. In case the altitude is not known, it may be found if the slant height and radius of base are known.

EXAMPLES FOR PRACTICE.

1. Find the total area of a cone having an altitude of 21 feet and a base 16 feet in diameter.

Ans. 765.75 square feet.

2. Find the volume of a cone if the area of the base is 12.566 square inches, and the slant height is $6\frac{1}{2}$ inches.

Ans. 25.898+ cubic inches.

3. The volume of a cone is 1,061.84 cubic inches and the altitude is 24 inches. What is the lateral area?

Ans. 507.64+ square inches.

FRUSTUMS OF PYRAMIDS.

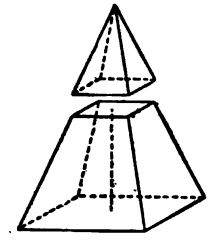
278. The frustum of a pyramid is the solid formed by passing a plane parallel to the base and cutting all the edges.

279. The upper plane of the frustum is called the *upper base* and the lower plane the *lower base.*

280. The *altitude* is the perpendicular distance between the bases.

281. To find the **lateral area** of the frustum, multiply one-half the sum of the two perimeters by the slant height.

282. Example. The lower base of a frustum of a square pyramid measures 12 inches on each side and the upper base measures 4 inches on a side; if the slant height is 7 inches, what is the lateral area?



FRUSTUM OF PYRAMID.

Perimeter of lower base $= 4 \times 12 = 48$ inches

Perimeter of upper base $= 4 \times 4 = 16$ inches

One-half the sum $= \frac{48 + 16}{2} = 32$ inches

The lateral area $= 7 \times 32 = 224$ square inches.

283. The total area is found by adding the areas of the bases to the lateral area.

In the above example we have:

Area of lower base $= 144$ square inches

Area of upper base $= 16$ square inches

Lateral area $= 224$ square inches

Total area $= 384$ square inches.

284. Volume. To find the volume of a frustum of a pyramid: (1) Find the areas of the two bases and add them together. (2) To this quantity (1) add the *square root* of the *product* of the two bases. (3) Multiply the sum thus obtained by one-third the altitude.

285. Example. The altitude of the frustum of a square pyramid is 10 feet. Each side of the lower base is 12 feet and each side of the upper base is 9 feet. What is the volume?

Area of lower base $= 12 \times 12 = 144$ square feet

Area of upper base $= 9 \times 9 = 81$ square feet

Sum (1) $= 225$ square feet

Square root of product (2) $= \sqrt{144 \times 81}$

Volume $= \frac{10}{3} \times (225 + 108)$

$= 1110$ cubic feet.

FRUSTUMS OF CONES.

286. The frustum of a cone is a solid formed by a plane parallel to the base and cutting the conical surface.

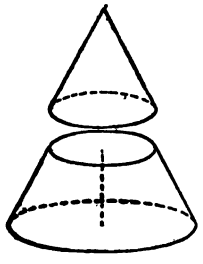
287. The upper plane is called the *upper base* and the lower plane the *lower base.*

288. The *altitude* is the perpendicular distance between the bases.

289. To find the **lateral area** of the frustum, multiply one-half the sum of the two perimeters by the slant height.

290. Example. The lower base of a frustum of a cone is 11 inches in diameter and the upper base is 8 inches in diameter. What is the convex area if the slant height is 5 inches?

$$\begin{aligned}
\text{Circumference of lower base} &= \pi\ d\ \text{(Art. 215)} \\
&= 3.1416 \times 11 \\
&= 34.5576 \text{ inches} \\
\text{Circumference of upper base} &= 3.1416 \times 8 \\
&= 25.1328 \text{ inches} \\
\frac{34.5576 + 25.1328}{2} &= 29.8452 \\
\text{Lateral area} &= 29.8452 \times 5 \\
&= 149.2260 \text{ square inches.}
\end{aligned}$$



FRUSTUM OF CONE.

291. The **total area** is equal to the areas of the bases added to the lateral area. Thus in the above example,

$$\begin{aligned}
\text{Area of upper base} &= \frac{\pi}{4}\ d^2\ \text{(Art. 215)} \\
&= .7854 \times 64 \\
&= 50.2656 \text{ square inches} \\
\text{Area of lower base} &= .7854 \times 121 \\
&= 95.0334 \text{ square inches} \\
&\quad 50.2656 \\
&\quad 95.0334 \\
&\quad 149.2260 \\
\text{Total area} &= 294.5250 \text{ square inches.}
\end{aligned}$$

292. The **volume** of the frustum is found in the same manner as the frustum of a pyramid.

(1.) Find the areas of the two bases and add them together. (2.) To this quantity (1) add the *square root* of the *product* of the two bases. (3.) Multiply the sum thus obtained by one-third the altitude.

293. Example. The altitude of the frustum of a cone is 18 inches. The upper base is 3 inches in radius and the lower base is 12 inches in diameter. What is the volume?

$$\begin{aligned}
\text{Area of lower base} &= 3.1416 \times 36 \\
&= 113.0976 \\
\text{Area of upper base} &= 3.1416 \times 9 \\
&= 28.2744 \\
\text{Sum (1)} &= 141.3720 \\
\text{Square root of product (2)} &= \sqrt{113.0976 \times 28.2744} \\
&= 56.5488 \\
\text{Volume} &= 6 \times (141.3720 + 56.5488) \\
&= 1{,}187.5248 \text{ cubic inches.}
\end{aligned}$$

EXAMPLES FOR PRACTICE.

1. Find the volume of the frustum of a cone having a slant height of 6 feet, an upper base 6 feet in diameter and a lower base 8 feet in diameter.

Ans. 228.973 cubic feet.

2. Find the lateral area of the frustum of a rectangular pyramid whose altitude is 8 inches and whose bases are 8×8 inches and 14×14 inches respectively.

Ans. 375.980 square inches.

3. What is the slant height of a frustum of a cone if the convex area is 502.656 square inches, the upper base 7 inches in diameter and the lower base 13 inches in diameter?

Ans. 16 inches.

SPHERE.

294. A sphere is a solid bounded by a curved surface, every point of which is equally distant from a point within called the center.

295. The *diameter* is a straight line drawn through the center and having its extremities in the curved surface. The *radius* is the straight line from the center to the surface; it is equal to one-half the diameter.

296. A sphere is *tangent* to a plane when the plane touches the sphere in only one point. A plane perpendicular to the extremity of a radius is tangent to the sphere.

297. Surface. To find the surface of a sphere, *multiply the square of the diameter by* 3.1416.

Example. Find the surface of a sphere which is 9 inches in diameter.

$9^2 = 81.$ $\quad 81 \times 3.1416 = 254.4696$ square inches.

298. To find the diameter when the surface is given, *divide the surface by* 3.1416 and extract the square root of the quotient.

Example. What is the diameter of a sphere whose area is 113.0976 square feet?

$$\frac{113.0976}{3.1416} = 36. \qquad \sqrt{36} = 6 \text{ feet.}$$

299. Volume. To find the volume of a sphere, *multiply the cube of the diameter by* .5236.

Example. A sphere is 10 inches in diameter; what is the volume?

$10^3 = 1000.$ $\quad 1000 \times .5236 = 523.6$ cubic inches.

Another example. Find the volume of a sphere having a diameter of 7 feet.

$7^3 = 343.$ $\quad 343 \times .5236 = 179.5948$ cubic feet.

300. To find the diameter when the volume is known, *divide the volume by* .5236 *and find the cube root of the quotient.*

Example. A certain sphere contains 8181.25 cubic feet; what is the diameter?

$$\frac{8181.25}{.5236} = 15625. \qquad \sqrt[3]{15625} = 25 \text{ feet.}$$

EXAMPLES FOR PRACTICE.

1. A sphere has an area of 153.9384 square inches. What is the volume?

Ans. 179.5948 cubic inches.

2. How much will a sphere of cast iron weigh if it is 8 inches in diameter, and cast iron weighs .26 pounds per cubic inch?

Ans. 3.6756 pounds.

3. How much will it cost to paint a sphere whose radius is $4\frac{1}{2}$ feet if it costs 25 cents to paint one square yard?

Ans. $7.07.

4. How much less surface has a sphere which is 12 inches in diameter than a sphere which has a volume of 1.7671 cubic feet?

Ans. 3.927 square feet.

5. What is the volume of a sphere if the area is 28.2744 square inches?

Ans. 14.1372 cubic inches.

A PRACTICAL PROBLEM.

To Find the Steam Space in a Horizontal Boiler.

Suppose the boiler represented in Fig. 11 has the following dimensions: length 12 feet, diameter 50 inches and the water level A B C D 16 inches below the top of the boiler.

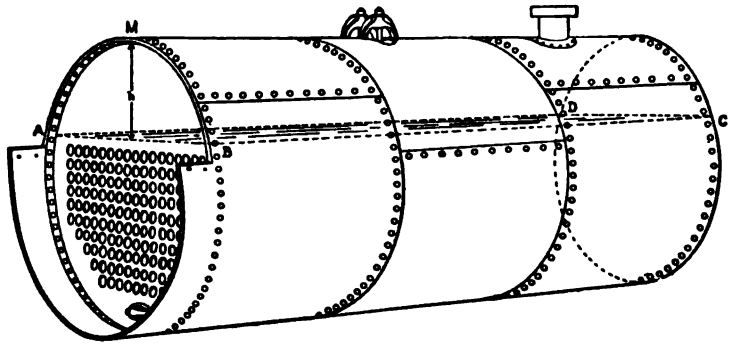


Fig. 11.

The steam space is equal to the volume above the plane A B C D which evidently is equal to the area A M B multiplied by the length of the boiler. To find the area A M B we may use the formula given in Art. 218 since A M B is a segment.

$$A = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}$$

In the above, $h = 16$ inches, and $D = 50$ inches. Substituting these values we have,

$$A = \frac{4 \times 16^2}{3}\sqrt{\frac{50}{16} - .608}$$
$$= 341.33\sqrt{3.125 - .608}$$
$$= 542.7 \text{ square inches.}$$

To reduce this area to square feet, divide by 144.

$$\frac{542.7}{144} = 3.768.$$

The volume equals this area multiplied by the length

$$3.768 \times 12 = 45.216 \text{ cubic feet.}$$

SECTION 5.

ALGEBRA.

PART I.

---

1. **Algebra** is that branch of mathematics in which we investigate the relations of numbers, abridging and generalizing the reasoning by means of symbols.

2. There are four kinds of **symbols** as follows:

1. Symbols of Quantity.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Abbreviation.

SYMBOLS OF QUANTITY.

3. **Symbols of Quantity** are usually the figures of arithmetic and the letters of the alphabet. If the quantity is a known quantity and is not given in figures, it is usually represented by such letters as a, b, c, etc., i. e., the first letters of the alphabet. If the quantities are unknown, they are represented by the last letters as x, y, z, etc.

4. Quantities closely related in the same problem are often designated by accents as a', a'', a''', etc. They are also designated by subscripts as a_1, a_2, a_3, etc. In the first case a is read a prime, etc., and in the second case a sub-one, etc.

SYMBOLS OF OPERATION.

5. The **signs** of addition, subtraction, multiplication and division are, as in arithmetic, $+$, $-$, $\times$, $\div$. The $\times$ sign is understood between figures unless the other signs are represented. These constitute, together with the following, the symbols of operation.

A **Coefficient** is a number prefixed to a quantity to show how many times that quantity is to be taken.

Thus in the quantity $7ax$, 7 is the coefficient and indicates that ax is to be taken 7 times, as follows: $ax+ax+ax+ax+ax$

$+ax+ax$. If no coefficient is expressed, the coefficient is understood to be 1. Thus in the quantity a, the coefficient is 1. In other words, when 1 is the coefficient, it is not written. If a number of factors are multiplied together, the product of any of them may be considered as the coefficient of those remaining. Thus in the quantity $abcd$, we may regard a as the coefficient of bcd, or ab as the coefficient of cd, abc as that of d, ac as that of bd, etc.

An **Exponent** is a figure or letter written above and to the right of a quantity to show how many times it is to be taken as a factor. Thus in the quantity a^3, the 3 indicates that a is to be taken 3 times as a factor, that is, $a \times a \times a$, or, as better written, aaa.

x^2 indicates xx and is read "x to the second power", or, more commonly "x square."

x^3 indicates xxx and is read "x to the third power", or "x cube".

x^4 indicates $xxxx$ and is read "x to the fourth power", or "x fourth".

When no exponent is written 1 is understood.

SYMBOLS OF RELATION.

6. The symbols of relation are the **Sign of Equality** and the **Sign of Inequality.**

The first is $=$ and is read "equals." The expression $x = a$ indicates that x and a are equal. Such a statement that two quantities are equal is termed an equation.

The second is $>$ or $<$ and is read "*is greater than*" or "*is less than*". The expressions $a > b$ and $a < b$ are read "a is greater than b" and "a is less than b" respectively.

SYMBOLS OF ABBREVIATION.

7. The sign of deduction is $\therefore$ and stands for "therefore" or "hence."

8. Parentheses, brackets, braces and the vinculum are termed the signs of aggregation, and indicate that the quantities enclosed by them are to be taken collectively. The use of these signs will be illustrated farther on.

9. The sign of continuation, standing for "and so on" or "continued according to the same law" is . . . . . Thus the expression a, $a+1$, $a+2$, $a+3$, ... is read "a, $a+1$, $a+2$, $a+3$, and so on".

ALGEBRAIC EXPRESSIONS.

10. An algebraic expression is a combination of algebraic symbols; such as $x + ax - b$.

11. A **term** is an algebraic expression, none of whose parts are separated by the + and — signs. In the preceding expression x, ax and b are terms.

Positive terms are preceded by a plus sign, as in the above expression x and ax. The + sign is called the positive sign. If no sign is expressed, the + sign is understood, as in the case of x which is the same as $+ x$.

Negative terms are those preceded by the minus sign, as $-b$. The — sign is called the negative sign. It must never be omitted. Equal terms with unlike signs neutralize or cancel each other: thus $a - a = 0$.

12. A **monomial** is an algebraic expression containing but one term as $4x$; a **polynomial** is an expression of more than one term, as $4x + y$, or $4x + a + 2b$. A polynomial of two terms, as $4x + y$, is called a **binomial.** One of three terms, as $4x + a + b$, is called a **trinomial.**

13. **Similar** or **like terms** are those differing only in numerical coefficient as $3xy$ and $11xy$. **Dissimilar** or **unlike terms** are those which are not similar, as ax^2y and axy^2.

14. The **degree** of a term is the number of literal (letter) factors contained in it. $7x$ is of the first degree, for it contains but one literal factor, x.

$3xy$ is of the second degree, for it contains two literal factors, x and y. a^3b^4 is of the seventh degree, for it contains seven literal factors, 3 a's and 4 b's. abc^3 is of the fifth degree, for it contains five literal factors, an a, a b, and 3 c's. **Homogeneous** terms are those of the same degree, as x^3 and ab^2, each of which is of the third degree.

15. The reciprocal of a quantity is 1 divided by that quantity. Thus the reciprocal of 3 is $\frac{1}{3}$ and the reciprocal of x is $\frac{1}{x}$.

SUBSTITUTION.

16. If we know the arithmetical value of the letter in an algebraic expression, we can find the numerical value of the expression by substituting the arithmetical values. Take, for instance, the following examples, where $a = 3$, $b = 2$ and $c = 4$;

$$a^2 + ab + c$$
substituting, $3^2 + 3 \times 2 + 4$
or, $9 + 6 + 4$ which is 19.

This would be ordinarily written thus,

$$a^2 + ab + c = 3^2 + 3 \times 2 + 4 = 9 + 6 + 4 = 19.$$

To illustrate fractions:

$$\frac{2a + ac}{b} = \frac{2 \times 3 + 3 \times 4}{2} = \frac{6 + 12}{2} = \frac{18}{2} = 9$$

As a further illustration:

$$a^2 + a\,(c - b) + \frac{a}{c} = 3^2 + 3\,(4 - 2) + \frac{3}{4}$$
$$= 27 + 3\,(2) + \frac{3}{4} = 27 + 3 \times 2 + \frac{3}{4}$$
$$= 27 + 6 + \frac{3}{4} = 33\,\frac{3}{4}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the numerical value of the following expressions, when $a = 2$, $b = 3$, $c = 1$, and $d = 4$:

1. $\frac{b^a}{a^d}$. Ans. $\frac{9}{16}$.
2. $3a^3 - 2a^2 b + c^3$. Ans. 1.
3. $b^d - a^b b^2$. Ans. 9.
4. $\frac{3d^2}{5ac} - \frac{2a}{3b^2}$. Ans. $\frac{628}{135}$.
5. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d}$. Ans. $\frac{47}{12}$.
6. $15a - 7\,(b + c) - d$. Ans. -2.
7. $25a^2 + 7\,(b^3 + c^2) + d^2$. Ans. 312.

ALGEBRAIC SOLUTION.

17. In order to illustrate the use of algebra in solving problems, let us take a few examples.

One number is 5 times as great as another. The sum of the two is 42. What are the numbers?

We will let the second number be represented by x. The first number was 5 times as great as the second, so the first number is $5 \times x$ or $5x$. The sum of the two is 42. Hence we may make the equation $5x + x = 42$ or $6x = 42$, and hence x must equal 7, and the numbers are 7 and 35.

The sum of two numbers is 47. Their difference is 23. What are the numbers?

If the sum of the two numbers is 47, we may call one x and the other $47 - x$, for $x + 47 - x = 47$. Now supposing x to be the lesser number, we may say $(47 - x) - x = 23$, or $47 - 2x = 23$, or $2x = 47 - 23 = 24$, or $x = 12$. Hence the greater number is $47 - 12$, or 35, and the lesser 12.

EXAMPLE FOR PRACTICE.

Divide 84 into three parts such that the second is twice the first, and the third is twice the second. Ans. 12, 24 and 48.

NEGATIVE QUANTITIES.

18. The signs + and — not only indicate the operations of addition and subtraction, but are also used to distinguish between quantities which are exactly reverse in quality or condition. For example, we may speak of a man's assets as + and his debts or liabilities as —. If we say a man's property is —$50, we mean that he is $50 in debt. The thermometer furnishes another illustration. The 0 is a certain fixed point above which quantities are called + and below which they are called —.

For example, what is the number of degrees difference between —8 and 29. The difference must be 8 + 29 or 37, because the 8 is on the opposite side of the 0 mark from the 29.

EXAMPLES FOR PRACTICE.

1. How many degrees difference between 47 and — 13. Ans. 60.
2. What is the difference between 47 and — 317. Ans. 364.

ADDITION.

19. **Addition** in Algebra, is the process of collecting two or more quantities into an equivalent expression called the **sum.** The sum of a, b, and c is $a + b + c$.

If either quantity is negative or a polynomial, it should be enclosed in a parenthesis. Thus the sum of a and $-b$ is $a + (-b)$, and the sum of $a - b$ and $c - d$ is $(a - b) + (c - d.)$

20. We see by Article 19 that adding a negative quantity is equivalent to substracting a positive quantity numerically equal to the negative quantity. For instance, if a man possesses \$600 and incurs a debt of \$50, his property is then \$600 + (— \$50) or \$600 — \$50 or \$550.

Adding a negative quantity is equal to subtracting a positive quantity numerically equal to it.

We may now see the connection by the illustration of the man incurring a debt. Suppose the man's property to be a, and that he incurs a debt of b, then we may write as before an expression of the man's property thus, $a + (-b) = a - b$.

We see by the above that we may add monomials by *uniting them with their respective signs*, and that the order in which the terms are united is immaterial. Thus the sum of a, $-b$, $-c$, and d, is $a - b - c + d$, or $d + a - b - c$, or $d - b + a - c$, and so on.

21. Suppose now that we wish to add $4x$ and $7x$. $4x$ means x taken 4 times, and $7x$ means x taken 7 times. Hence $4x + 7x$ means x taken $4 + 7$, or 11 times, or $11x$.

If the quantities were $-4x$ and $7x$, we should, instead of adding 4 to 7, decrease 7 by 4, thus obtaining 3. Hence our answer would be $7x - 4x$ or $3x$. This may be seen more clearly by dividing the $7x$ into $3x + 4x$ and adding the $-4x$ thus, $3x + 4x - 4x$. The $4x$ and $-4x$ being opposite in sign cancel each other, leaving $3x$.

We may comprehend the above a little more clearly by observing the following:

A man has 7 dollars and receives 4 dollars more. Therefore he has 11 dollars. In the former consideration we had x in the place of dollars, the x standing for whatever we might desire it

to stand. If a man has a debt of 7 dollars and incurs another of 4 dollars, he has a total debt of 11 dollars. His assets are, therefore, -11 dollars, i. e., his property is a negative quantity.

If a man has 7 dollars and incurs a debt of 4, his assets are $7 + (-4)$ or $7 - 4$ or 3 dollars. If he had 4 dollars and incurred a debt of 7, he would have $4 + (-7)$ or $-7 + 4$ whichever way we choose to express it. Therefore his assets would be -3 dollars, i. e., he would have a debt of 3 dollars above what he was able to pay.

EXAMPLES FOR PRACTICE.

Add the following:

1. 11 and -5. Ans. 6.
2. -4 and -7. Ans. -11
3. $12mn$ and $-19mn$. Ans. $-7mn$.
4. $-13abc$ and $22abc$. Ans. $9abc$.
5. $7a$, $-a$ and $-3a$. Ans. $3a$.
6. $-6m$, m, $-11m$ and $5m$. Ans. $-11m$.
7. $13ab$, $-7ab$, $-ab$ and $-6ab$. Ans. $-ab$.
8. $5ax$, $-11b$, $-ax$ and $6b$. Ans. $4ax - 5b$.
9. $5m$, $-3n$, n, $-2m$, $-b$, and $3n$. Ans. $3m + n - b$.
10. $3x$, $-y$, $-x$, 6, $-8y$, $-2x$, $4y$ and -5. Ans. $-5y + 1$.

22. If polynomials are to be added, each polynomial is regarded as the sum of its monomial terms. Thus $a + c - bx$ may be regarded as the sum of a, c and $-bx$.

Suppose that it is required to add $6a + 2b$ and $3a - b$. We may simply unite their terms with their respective signs, thus: $6a + 2b + 3a - b$, which equals $9a + b$.

It is more convenient to write similar terms in the same vertical column, and add, thus:

$$\begin{array}{r} 6a + 2b \\ 3a - b \\ \hline 9a + b \end{array}$$

The following furnishes another illustration. Find the sum of $7a + 4bc + 8ax$, $a + 2bc - 3ax$ and $7bc - 4a$.

$$\begin{array}{r} 7a + 4bc + 8ax \\ a + 2bc - 3ax \\ -4a + 7bc \\ \hline 4a + 13bc + 5ax \end{array}$$

From the preceding is derived the following rule:

To add two or more expressions, write them one beneath the other, similar terms in the same vertical columns. Find the sum of the terms in each column, and write the results with their respective signs.

EXAMPLES FOR PRACTICE.

Find the sum of the following:

1. $5mn^2 - 8x^2y$, $-mn^2 + x^2y$, $-6mn^2 - 3x^2y$, $4mn^2 + 7x^2y$, $2mn^2 + 3x^2y$, and $-mn^2 - 2x^2y$. Ans. $3mn^2 - 2x^2y$.

2. $2a - 5b - c + 7$, $3b - 2 - 6a + 8c$, $c + 3a - 4$, and $1 + 2b - 5c$. Ans. $-a + 3c + 2$.

3. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$. Ans. $x - y + 3m + 3n$.

4. $3x - 2y - z$, $3y - 5x - 7z$, $8z - y - x$, and $4x$. Ans. x.

5. $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4p$. Ans. $6mn - ab - 4c + 3x + 3m^2 - 4p$.

6. $3a + b - 10$, $c - d - a$, $-4c + 2a - 3b - 7$, and $4x^2 + 5 - 18m$. Ans. $4a - 2b - 12 - 3c - d + 4x^2 - 18m$.

SUBTRACTION.

23. **Subtraction** is the process of taking one quantity from another.

The *subtrahend* is the quantity to be subtracted.

The *minuend* is the quantity from which it is to be subtracted.

The *remainder* is the result of the operation.

NOTE. The minuend is evidently equal to the sum of the subtrahend and remainder.

24. Subtracting a quantity being the opposite of addition, is equivalent to adding that quantity with the opposite sign.

For example, subtract b from a and we have $a - b$, subtract $-b$ from a and we have $a + b$ as a result.

Hence we may say that, to subtract a quantity, we *add that quantity with the sign changed.*

Let us take a few examples: if we subtract $7x$ from $2x$ we have $2x - 7x$ or $-5x$ as a result. If we subtract $-7x$ from $2x$

we have $2x + 7x$ or $9x$. If we subtract $-7x$ from $-2x$ we have $-2x + 7x$ or $5x$, and so on.

EXAMPLES FOR PRACTICE.

1. Subtract $-2a$ from $3a$. Ans. $5a$.
2. Subtract $3a$ from $-2a$. Ans. $-5a$.
3. Subtract $-5a$ from $-2a$. Ans. $3a$.
4. Subtract $-6b$ from $2b$. Ans. $8b$.
5. Subtract $8c$ from $4c$. Ans. $-4c$.
6. Subtract $4c$ from $8c$. Ans. $4c$.

If we are dealing with polynomials, we follow the same plan of adding each term of the subtrahend with its sign changed. Similar terms should be placed in the same vertical columns.

The following examples illustrate this:

Subtract $6ab + 4x^2 - y$ from $9ab - 3x^2 + 4y$.

$$\begin{array}{r} 9ab - 3x^2 + 4y \\ 6ab + 4x^2 - y \\ \hline 3ab - 7x^2 + 5y \end{array}$$

The answer is, as we see, obtained by adding the terms of the subtrahend with their signs changed. This is further illustrated by the following example:

$$\begin{array}{r} 46abx - 17x^2y \\ 13abx - 11x^2y \\ \hline 33abx - 6x^2y \end{array}$$

EXAMPLES FOR PRACTICE.

1. Subtract $a - b + c$ from $a + b - c$. Ans. $2b - 2c$.
2. From $31x^2 - 3y^2 + ab$ take $17x^2 + 5y^2 - 4ab + 7$.
   Ans. $14x^2 - 8y^2 + 5ab - 7$.
3. Take $4a - b + 2c - 5d$ from $d - 3b + a - c$.
   Ans. $-3a - 2b - 3c + 6d$.
4. From $a + b$ take $2a - 2b$ and $-a + b$. Ans. $2b$.
5. From $a - b - c$ take $-a + b + c$ and $a - b + c$,
   Ans. $a - b - 3c$.

USE OF PARENTHESES

25. As parentheses are often used it is necessary to understand how to remove and introduce them.

The expression

$$a + 3bc + (9bc + 5)$$

indicates that $9bc + 5$ is to be added to $a + 3bc$, hence its value is $a + 3bc + 9bc + 5$, or $a + 12bc + 5$.

Again the expression

$$a + 3bc - (9bc + 5)$$

indicates that $9bc + 5$ is to be subtracted from $a + 3bc$, giving, by changing the signs of the subtrahend, $a + 3bc - 9bc - 5$, or $a - 6bc - 5$.

It will be observed in the first case, where the parenthesis is preceded by a + sign, that, after removal of the parenthesis, the signs within the parentheses are the same as before removal. In the second case, however, where the parenthesis is preceded by the — sign, all the terms within the parenthesis have, after its removal, opposite signs.

We may, therefore, formulate the following rules:

A parenthesis preceded by a + sign may be removed without altering the signs of the enclosed terms.

A parenthesis preceded by a — sign may be removed if the sign of each enclosed term be changed from + to —, or from — to +.

If one pair of parentheses encloses another, the innermost pair should be removed first, and then the other.

To illustrate:

$$2a + 9bc - (4d + 3) + (8a - bc).$$

Removing the parentheses

$$2a + 9bc - 4d - 3 + 8a - bc.$$

Collecting we have

$$10a + 8bc - 4d - 3.$$

Simplify the following expression:

$$9a + 7b - (3a - \overline{a + b})$$

Removing the vinculum

$$9a + 7b - (3a - a - b$$

Removing the parenthesis

$$9a + 7b - 3a + a + b$$

Collecting, we have,

$$7a + 8b.$$

EXAMPLES FOR PRACTICE.

Simplify the following:

1. $3a - [2a - (a + 2)]$. Ans. $2a + 2$.
2. $5x - (2x - 3y) - (2x + 4y.)$ Ans. $x - y$.
3. $m^3 - 2n + (a - n + 3m^3) - 5a - 3n + m^3$.
   Ans. $5m^3 - 6n - 4a$.
4. $3x - (5x - [4x - y - \overline{x}]) - (x - 3y)$.
   Ans. $2x + 2y$.

26. To enclose any number of terms in a parenthesis, we use the converse of the rule for removing a parenthesis.

Any number of terms may be enclosed in a parenthesis preceded by a + sign without changing their signs.

Any number of terms may be enclosed in a parenthesis preceded by a — sign, if the sign of each term is changed from + to —, or from — to +.

Enclose the last three terms of $a + b - c - d + e$

$$a + b - (c + d - e)$$

EXAMPLES FOR PRACTICE.

Enclose the last three terms of each of the following expressions in a parenthesis preceded by a — sign.

1. $3a - 2b + 5c - 4d$. Ans. $3a - (2b - 5c + 4d)$.
2. $a - b - c + d + e - f$. Ans. $a - b - c - (-d - e + f)$.
3. $m^3 + 5m^2 - 6m + 3$. Ans. $m^3 - (-5m^2 + 6m - 3)$.
4. $a^2 - b^2 - c^2 + 2ab + 2ac$. Ans. $a^2 - b^2 - (c^2 - 2ab - 2ac)$.

MULTIPLICATION.

27. Instead of adding the same quantities several times, we use a shorter process called **multiplication.** This is the process of taking one quantity as many times as there are units in another. The multiplication of a by 4 means that a is added 4 times, thus: $a + a + a + a$ or $4a$; but we abridge this process thus: $4 \times a = 4a$. If we multiply x by a we have a similar result, $x \times a = ax$.

The *multiplicand* is the quantity to be multiplied.

The *multiplier* is the quantity which shows how many times the multiplicand is to be taken.

The *product* is the result of the process.

NOTE. The multiplicand and the multiplier are called *factors*.

28. As in arithmetic, the product of two or more numbers is the same regardless of the order in which the numbers are taken. Thus $7 \times 3 = 21$ and $3 \times 7 = 21$, and $2 \times 6 \times 9 = 108$, $6 \times 9 \times 2 = 108$ and $9 \times 6 \times 2 = 108$. Similarly in algebra, $a \times b = ab$ and $b \times a = ab$, and $a \times b \times c = abc$, and $b \times c \times a = abc$.

29. Suppose, now, that in arithmetic we wished to multiply 17 by 4, we should have $4 \times 17 = 68$. We could obtain the result by multiplying the 17 expressed as two quantities, as 12 and 5. For example:

$$\begin{array}{r} 12 + 5 \\ \hline 4 \\ \hline 48 + 20 \end{array}$$

and $48 + 20 = 68$.

Again we might express 17 thus: $23 - 6$ and multiply.

$$\begin{array}{r} 23 - 6 \\ \hline 4 \\ \hline 92 - 24 \end{array}$$

and $92 - 24 = 68$.

Suppose, now, in algebra, we wished to multiply $a - b$ by x, we should have

$$\begin{array}{r} a - b \\ \hline x \\ \hline ax - bx \end{array}$$

Thus we may pursue the plan of dividing the multiplicand and multiplying for 4 multiplied by 17 is 68, and 4 multiplied by $23 - 6$ is $92 - 24$ or 68, for in this case we have taken 23 times the quantity and subtracted 6 times the quantity, which is the same as taking the quantity 17 times. Similarly in algebra, we multiply x by $a - b$, obtaining $ax - bx$.

30. Now both the multiplicand and multiplier may be separated into parts. Take $a + b$ and $c - d$. We may obtain the product by taking the product of $a + b$ and c and subtracting the product of $a + b$ and d, thus: $ac + bc - (ad + bd)$ or $ac + bc - ad - bd$. Let us take a numerical example. $15 \times 11 = 165$, 15 equals $17 - 2$ and 11 equals $7 + 4$. Now the product of $17 - 2$ and $7 + 4$ is $7 \times (17 - 2) + 4 \times (17 - 2)$, or $119 - 14 + 68 - 8$ or 165.

31. We have seen that if we multiply $a + b$ and $c - d$, we have $ac + bc - ad - bd$. From this we see that when a positive quantity is multiplied by a positive quantity the product is a positive quantity, and that when a positive quantity is multiplied by a negative quantity the result is a negative quantity. If a negative quantity is multiplied by a positive quantity, the result is negative, i. e., opposite to that when a positive is multiplied by a positive quantity. Again, a negative quantity gives an opposite result when multiplied by a negative quantity, i. e., a positive quantity.

We may enumerate this thus:

\+ multiplied by + produces +
\+ multiplied by — produces —
— multiplied by + produces —
— multiplied by — produces +

We may state this more concisely by saying that *like signs produce* + *and unlike signs produce* —.

32. Suppose that the product of $4a$ and $7b$ is required. As the factors may be written in any order, we may write

$$4a \times 7b = 4 \times 7 \times a \times b = 28ab.$$

We may say, therefore, that *the coefficient of the product is the product of the coefficient of the factors.*

Let us now take the case of x^4 and x^2:

$$x^4 = x \times x \times x \times x$$
$$\text{and } x^2 = x \times x$$

Hence $x^4 \times x^2 = x \times x \times x \times x \times x \times x = x^6$,

and we may say that *the exponent of a letter in the product is the sum* of its exponents in the factors.

Thus

$$x^4 \times x^3 \times x^5 = x^{4+3+5} = x^{12}.$$

We now arrive at the following rule for monomials.

Write the product of the coefficient, and after it that of the literal quantities, giving to each letter an exponent equal to the sum of its exponents in the factors. Make the product — when there is an odd number (i. e., not divisible by 2) of — signs in the factors.

The following example illustrates the rule:

Find the product of $3ax^2$, $13y$, and $-7a^3$.

$$-273a^3x^2y.$$

273 is the product of 3, 13 and 7. a^3 is the product of a and a^2, while x and y occur as the other factors.

EXAMPLES FOR PRACTICE.

Multiply

1. $15m^6n^6$ and $3mn$. Ans. $45m^6n^7$.
2. $17abc$ and $-8abc$. Ans. $-136a^2b^2c^2$.
3. $-12a^2x$ and $-2x^2y$. Ans. $24a^2x^3y$.
4. $2a^m - b^n$ and $5a^2b$. Ans. $10a^{m+2}b - 5a^2b^{n+1}$.
5. $-7m^nx^2$ and $m^nx^ny^2$. Ans. $-7m^{2n}x^{n+2}y^2$.
6. $-6ab^2c$ and a^2bm. Ans. $-6a^3b^2cm$.

33. Suppose now we wish to multiply a polynomial by a monomial. We know that by multiplying $a - b$ by x we have $ax - bx$. Hence we may formulate the following rule for multiplying a polynomial by a monomial.

Multiply each term of the multiplicand by the multiplier and connect the results by their proper signs.

Let us find the product of multiplying $3x - y$ by $2xy$:

$$\begin{array}{r} 3x - y \\ 2xy \\ \hline 6x^2y - 2xy^2 \end{array}$$

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $x^2 - 2x - 3$ by $4x$. Ans. $4x^3 - 8x^2 - 12x$.
2. $3x^2 + 6x - 7$ by $-2x^2$. Ans. $-6x^5 - 12x^4 + 14x^2$.
3. $3m^2 - 5mn - n^2$ by $-2m$. Ans. $-6m^3 + 10m^2n + 2mn^2$.
4. $5 - 6a - 8a^2$ by $-6a^2$. Ans. $-30a^2 + 36a^3 + 48a^4$.
5. $a^3 - 3a^2b + 3ab^2 - b^3$ by a^2b^2.

Ans. $a^5b^2 - 3a^4b^3 + 3a^3b^4 - a^2b^5$.

34. If we wish to multiply a polynomial by a polynomial, we see, by observing the example of multiplying $a + b$ and $c - d$ which has been given already as equal to $ac + bc - ad - bd$, that to obtain the product we should

Multiply each term of the multiplicand by each term of the multiple and add the partial products.

Multiply $3a - 2b$ by $2a - 5b$.

$$\begin{array}{r} 3a - 2b \\ 2a - 5b \\ \hline 6a^2 - 4ab \\ \quad - 15ab + 10b^2 \\ \hline 6a^2 - 19ab + 10b^2 \end{array}$$

Multiply $6ab - 8b^2 + 4a^2$ by $-4b^2 + 2a^2 - 5ab$. First arrange according to ascending powers of a, thus:

$$\begin{array}{r} 4a^2 + 6ab - 8b^2 \\ 2a^2 - 3ab - 4b^2 \\ \hline 8a^4 + 12a^3b - 16a^2b^2 \\ \quad - 12a^3b - 18a^2b^2 + 24ab^3 \\ \quad\quad - 16a^2b^2 - 24ab^3 + 32b^4 \\ \hline 8a^4 \quad - 50a^2b^2 \quad + 32b^4. \end{array}$$ Ans.

Multiply $1 - x + x^2 - x^3$ by $1 + x$.

$$\begin{array}{r} 1 - x + x^2 - x^3 \\ 1 + x \\ \hline 1 - x + x^2 - x^3 \\ \quad x - x^2 + x^3 - x^4. \\ \hline 1 \quad - x^4. \end{array}$$ Ans.

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $3x + 2$ and $5x - 7$. Ans. $15x^2 - 11x - 14$.
2. $a^2 + ab + b^2$ and $b - a$. Ans. $b^3 - a^3$.
3. $1 + x + x^2 + x^3$ and $ax - a$. Ans. $ax^4 - a$.
4. $m^2 - mn - 3n^2$ and $2m^2 - 6mn$. Ans. $2m^4 - 8m^3n + 18mn^3$.

5. $5a^3 + 4b^2 - 3ab$ and $6a - 5b$.

Ans. $30a^3 - 43ab^2 + 39ab^2 - 20b^3$.

6. $2x^2 - 3x + 5$ and $x^2 + x - 1$.

Ans. $2x^4 - x^3 + 8x - 5$.

7. $a^3b - a^2b^2 - 4ab^3$ and $2a^2b - ab^2$.

Ans. $2a^5b^2 - 3a^4b^3 - 7a^3b^4 + 4a^2b^5$.

8. $x^2 + y^2 - xy$ and $xy + y^2 + x^2$.

Ans. $x^4 + x^2y^2 + y^4$.

9. $m^4 - m^3n + m^2n^2 - mn^3 + n^4$ and $m^2 - 2mn - 3n^2$.

Ans. $m^6 - 3m^5n + mn^5 - 3n^6$.

10. $x^{m+2}y - 3xy^{n-1}$ and $4x^{m+5}y^2 - 4x^4y^n$.

Ans. $4x^{2m+7}y^3 - 16x^{m+6}y^{n+1} + 12x^5y^{2n-1}$.

11. $2a^3 - 5a^2 - 6a + 4$ and $4a^3 + 10a^2 - 12a - 8$.

Ans. $8a^6 - 98a^4 + 152a^2 - 32$.

12. $a^3 - 3a^2b + 3ab^2 - b^3$ and $a^2 - 2ab + b^2$.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

Find the product of the following :

13. $x - 3$, $x + 4$, and $x - 7$.

Ans. $x^3 - 6x^2 - 19x + 84$.

14. $a + b$, $a^2 - ab + b^2$, and $a^3 - b^3$. Ans. $a^6 - b^6$.

15. $2m - 1$, $3m + 4$, and $6m - 5$.

Ans. $36m^3 - 49m + 20$.

16. $x + 1$, $3x - 2$, and $3x^2 - x - 2$.

Ans. $9x^4 - 13x^2 + 4$.

17. $2x - 1$, $3x + 2$, $4x - 3$, and $5x + 4$.

Ans. $120x^4 + 26x^3 - 111x^2 - 14x + 24$.

DIVISION.

35. **Division** is, in algebra, the process of finding one of two factors, when their product and the other factor are given.

Division is the converse of multiplication.

36. Division is indicated by the sign $\div$, but is more commonly expressed by a line, the quantity to be divided being above and the quantity by which it is to be divided being below.

Thus $\frac{16a^2x}{4a}$ means $16a^2x$ divided by $4a$ or $4ax$.

The *dividend* is the product of the factors.

The *divisor* is the given factor.

The *quotient* is the required factor.

37. Since the dividend is the product of the divisor and quotient, it follows that:

If the divisor is + and the quotient is +, the dividend is +.

If the divisor is — and the quotient is +, the dividend is —.

If the divisor is + and the quotient is —, the dividend is —.

If the divisor is — and the quotient is —, the dividend is +.

In other words, if the dividend and divisor have like signs, the quotient is +, and if the dividend and divisor have unlike signs, the quotient is —, or as in multiplication *like signs produce* +, *and unlike signs produce* —.

If we have for example $\frac{16a^2x}{4a}$ the quotient is $4ax$ and if we have $\frac{-16a^2x}{-4a}$ the quotient is $4ax$, but if we have $\frac{-16a^2x}{4a}$ or $\frac{16a^2x}{-4a}$ the quotient is $-4ax$.

NOTE. In dividing $16a^2x$ by $4a$ we must remember that a^2 is $a \times a$, and that the a below the line cancels one of the a's above it, leaving a single a in the quotient.

38. The preceding note brings us to a consideration of exponents in division. Suppose our problem be to divide a^7 by a^4, in other form $\frac{a^7}{a^4}$. As the product of divisor and quotient must equal the dividend, our problem becomes to determine what quantity multiplied by a^4 will equal a^7. This quantity will evidently be a^3 as is apparent from the following:

$$\underbrace{a \times a \times a \times a}_{a^4} \times \underbrace{a \times a \times a}_{a^3} = a^7$$

That is, the exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

Let us take a further illustration:

$$\frac{a^p}{a^q} = a^{p-q}$$

p and q represent any quantities.

We now have the following rule for the division of monomials:

Write the quotient of the coefficients and annex to it the literal quantities, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Take the sign of quotient + when the dividend and divisor have like signs, and — when they have unlike signs.

Divide $54b^7$ by $-9b^4$.

By the above rule, $\frac{54\,b^7}{-9b^4} = -6b^{7-4} - 6b^3$.

EXAMPLES FOR PRACTICE.

Divide in each example.

1. 84 by -12. Ans. -7.
2. -343 by 7. Ans. -49.
3. $12a^5$ by $4a$. Ans. $3a^4$.
4. $2m^3n^4$ by $-mn^3$. Ans. $-2m^3n$.
5. $-8x^5y^3$ by $-4x^2$. Ans. $2y^3$.
6. $72m^4n$ by $-12m^3$. Ans. $-6m^3n$.
7. $18a^mb$ by $6ab$. Ans. $3a^{m-1}$.
8. $a^{m+n}b^{m+n}$ by a^mb^n. Ans. a^nb^m.
9. $-91x^4y^3z^2$ by $-13x^2y^3$. Ans. $7x^2yz^2$.
10. $18m^3n^4p^5$ by $-2m^2np^5$. Ans. $-9mn^3$.

39. If a polynomial is to be divided by a monomial we have simply the converse of Article 33. There we had each term of the multiplicand multiplied by the multiplier. Here we have each term of the dividend divided by the divisor. Hence the rule:

Divide each term of the dividend by the divisor, and connect the results by their proper signs.

Divide $9a^3b - 6a^4c + 12a^2bc$ by $-3a^2$.

By this rule,

$$\frac{9a^3b - 6a^4c + 12a^2bc}{-3a^2} = -3ab + 2a^2c^2 - 4bc. \quad \text{Ans.}$$

Divide the following:

1. $9x^4 + 27x^3 - 21x^2$ by $-3x^2$. Ans. $-3x^2 - 9x + 7$.
2. $30a^3 - 75a^4b$ by $15a^3$. Ans. $2 - 5ab$.
3. $2x^3y^2z - 12xy^2z^3$ by $-2xy^2z$. Ans. $-x^2 + 6z^2$.
4. $5a^2bc - 5ab^2c + 5abc^2$ by $-5abc$. Ans. $-a + b - c$.
5. $20x^4 - 12x^3 - 28x$ by $4x$. Ans. $5x^3 - 3x - 7$.
6. $9a^3bc - 3a^2b + 18a^2bc$ by $-3ab$. Ans. $-3a^4c + a - 6a^2c$.
7. $15x^my^nz^r - 35x^{m+s}y^{r+n}z$ by $5x^my^nz$. Ans. $3z^{r-1} - 7x^sy^r$.

40. If a polynomial is to be divided by a polynomial the dividend and divisor are arranged according to the descending powers of x. Suppose we wish to divide $12 + 10x^3 - 11x - 21x^2$ by $2x^2 - 4 - 3x$. In this case we wish to find a quantity which, when multiplied by the divisor, $2x^2 - 3x - 4$, will give $10x^3 - 21x^2 - 11x + 12$. It is evident that some term multiplied by $2x^2$ will give $10x^3$. Hence $10x^3$ is the product of the highest powers of x in both divisor and quotient. The first term of the quotient, therefore, is $10x^3$ divided by $2x^2$ or $5x$.

The divisor multiplied by $5x$ gives the product of $10x^3 - 15x^2 - 20x$, which, when subtracted from the dividend, leaves the remainder $-6x^2 + 9x + 12$. The rest of the quotient multiplied by the divisor must give this remainder as a product. So to obtain the next term of the quotient we proceed as before with the remainder as a new dividend. The term containing the highest power of x in the dividend divided by that of the divisor gives -3 as the second term of the quotient. The divisor multiplied by -3 gives a product $-6x^2 + 9x + 12$. This product subtracted from the new dividend leaves no remainder. The required quotient is therefore $5x - 3$.

It is convenient to arrange the work as follows:

$10x^3 - 21x^2 - 11x + 12$	$2x^2 - 3x - 4$	Divisor.
$10x^3 - 15x^2 - 20x$	$5x - 3$	Quotient.
$-6x^2 + 9x + 12$		
$-6x^2 + 9x + 12$		

The following is a good rule for the division of polynomials:

Arrange the dividend and divisor with respect to descending powers of some common letter.

To obtain the first term of the quotient, divide the first term of the dividend by the first term of the divisor.

Multiply the entire divisor by this term, and subtract the product from the dividend, arranging the remainder in the same order of powers as the dividend and divisor.

With the remainder as a new dividend, proceed as before and continue until there is no remainder.

Divide $21x^2y^2 - 22xy - 8$ by $3xy - 4$.

$$\begin{array}{r|l} 21x^2y^2 - 22xy - 8 & 3xy - 4 \\ \cline{2-2} 21x^2y^2 - 28xy & 7xy + 2, \text{ Ans.} \\ \cline{1-1} 6xy - 8 & \\ 6xy - 8 & \\ \cline{1-1} \end{array}$$

Divide $9ab^2 + a^3 - 9b^3 - 5a^2b$ by $3b^2 + a^2 - 2ab$.

$$\begin{array}{r} a^2 - 2ab + 3b^2) \; a^3 - 5a^2b + 9ab^2 - 9b^3 \; (a - 3b, \text{ Ans.} \\ a^3 - 2a^2b + 3ab^2 \\ \hline - 3a^2b + 6ab^2 - 9b^3 \\ - 3a^2b + 6ab^2 - 9b^3 \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

Divide the following:

1. $6x^2 - x - 35$ by $3x + 7$. Ans. $2x - 5$.
2. $a^2 - 4ab + 4b^2$ by $a - 2b$. Ans. $a - 2b$.
3. $3ab^2 + 3b^3 - 4a^2b - 4a^3$ by $a + b$. Ans. $3b^2 - 4a^2$.
4. $8m^2 + 35 - 36m$ by $5 + 2m$. Ans. $4m^2 - 10m + 7$.
5. $16m^4 - 1$ by $2m - 1$. Ans. $8m^3 + 4m^2 + 2m + 1$.
6. $a^2 - b^2 + c^2 - 2ac$ by $a + b - c$. Ans. $a - b - c$.
7. $x^4 + y^4 + x^2y^2$ by $x^2 + y^2 + xy$. Ans. $x^2 - xy + y^2$.
8. $x^5y - xy^5$ by $x^3 + y^3 + xy^2 + x^2y$. Ans. $x^2y - xy^2$.
9. $x^6 - 6x^4 + 5x^2 - 1$ by $x^3 + 2x^2 - x - 1$. Ans. $x^3 - 2x^2 - x + 1$.

FORMULAE.

41. A *formula* is the algebraic expression of a general rule.

By means of the following, we are able to abridge much algebraic work:

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array} \qquad \begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array} \qquad \begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad - ab - b^2 \\ \hline a^2 \quad - b^2 \end{array}$$

In the first case we have $(a + b)^2 = a^2 + 2ab + b^2$. In the expression $a + b$, a is the first and b the second quantity; a^2 is the square of a, ab is the product of a and b, and b^2 the square of b, hence we may say that:

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.

In the second case we have $(a - b) = a^2 - 2ab + b^2$, which relation may be stated thus:

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.

In the third case we have $(a + b)\ (a - b) = a^2 - b^2$, or

The product of the sum and difference of two quantities is equal to the difference of their squares.

In the preceding consideration a and b simply stand for any two quantities, and the laws derived by their use are applicable to any two quantities with which we deal.

Let us observe the above with numerical values for a and b. Let $a = 3$ and $b = 2$.

Then:

$(3 + 2)^2 = 9 + 2 \times 6 + 4 = 25$, and as we know $3 + 2 = 5$ and $5^2 = 25$.

Again:

$(3 - 2)^2 = 9 - 2 \times 6 + 4 = 1$ and $3 - 2 = 1$ and $1^2 = 1$.

And

$(3+2)\ (3 - 2) = 9 - 4 = 5$ and $3+2 = 5$ and $3 - 2 = 1$ and $5 \times 1 = 5$.

The above would, of course, hold true for any values which might be assigned to a and b.

EXAMPLES FOR PRACTICE.

Write by inspection the values of the following:

1. $(x-4)^2$.
2. $(7-2x)^2$.
3. $(2m+3n)^2$.
4. $(x^4-y^2)^2$.
5. $(3x+11)\ (3x-11)$.
6. $(a^m+a^n)\ (a^m-a^n)$.
7. $(10x^3+9x^2)^2$.

Expand,

8. $(x+y+z)\ (x-y-z)$.
9. $(1+a-b)\ (1-a+b)$.

We find by multiplication:

$$\begin{array}{r} x+5 \\ x+3 \\ \hline x^2+5x \\ \quad +3x+15 \\ \hline x^2+8x+15 \end{array} \qquad \begin{array}{r} x-5 \\ x-3 \\ \hline x^2-5x \\ \quad -3x+15 \\ \hline x^2-8x+15 \end{array}$$

From the previous products we may deduce the following laws:

I. The coefficient of x is the algebraic sum of the numbers in the factors.

II. The last term is the product of the numbers.

With the aid of these laws the product of two binomials may be written by inspection.

EXAMPLES FOR PRACTICE.

Write by inspection the following:

1. $(x+7)\ (x+5)$.
2. $(x+8)\ (x-2)$.
3. $(x-3)\ (x+1)$.
4. $(x+1)\ (x+12)$.
5. $(x-8)\ (x-6)$.
6. $(x+9)\ (x-5)$.
7. $(x^2-3)\ (x^2-7)$.
8. $(x^3+2a)\ (x^3-6a)$.

42. The following results may be verified by division:

(1). $\dfrac{a^2-b^2}{a+b}=a-b$.

(2). $\dfrac{a^2-b^2}{a-b}=a+b$.

(3). $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$.

(4). $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$

Formulas (3) and (4) may be stated as follows:

If the sum of the cubes of two quantities be divided by the sum of those quantities, the quotient is equal to the square of the first quantity, minus the product of the two, plus the square of the second.

If the difference of the cubes of two quantities be divided by the difference of those quantities, the quotient is equal to the square of the first quantity, plus the product of the two, plus the square of the second.

EXAMPLES FOR PRACTICE.

Write by inspection the values of the following:

1. $\frac{x^2 - 81}{x - 9}$
2. $\frac{x^2 + 1}{x + 1}$
3. $\frac{125 + 64a^3}{5 + 4a}$
4. $\frac{a^3 - 8}{a - 2}$
5. $\frac{x^3 - 64}{x - 4}$
6. $\frac{a^3 + 343}{a + 7}$
7. $\frac{x^6 - y^6}{x^2 - y^2}$
8. $\frac{x^3 + 125y^3}{x + 5y}$

Divide the following:

9. $27x^3y^3 - 64z^3$ by $3xy - 4z$. Ans. $9x^2y^2 + 12xyz + 16z^2$.
10. $64m^3 - 216n^6$ by $4m - 6n^2$. Ans. $16m^2 + 24mn^2 + 36n^4$.
11. $343 + 125x^3y^3$ by $7 + 5xy$. Ans. $49 - 35xy + 25x^2y^2$.

By dividing in each of the following cases we get:

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3.$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3.$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

From these results we may deduce the following laws:

I. The number of terms is the same as the exponent of a in the dividend.

II. The exponent of a in the first term is less by 1 than the exponent of a in the dividend, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. When the divisor is $a - b$ the terms in the quotient are all positive and alternately positive and negative when the divisor is $a + b$.

The following principles are of great importance in connection with the division of binomials:

(1.) $a^n + b^n$ is divisible by $a+b$ if n is odd, and by neither $a+b$ nor $a-b$ if n is even.

(2.) $a^n - b^n$ is divisible by $a - b$ if n is odd, and by both $a+b$ and $a-b$ if n is even.

Divide $a^7 - b^7$ by $a - b$.

Applying the laws just stated, we have:

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6, \quad \text{Ans.}$$

Divide $x^4 - 81$ by $x+3$.

Since $81 = 3^4$, we have

$$\frac{x^4 - 3^4}{x + 3} = x^3 - 3x^2 + 9x - 27, \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

By inspection, write the values of the following:

1. $\dfrac{a^6 - b^6}{a - b}$
2. $\dfrac{m^7 + n^7}{m + n}$
3. $\dfrac{1 - x^4}{1 - x}$
4. $\dfrac{x^4 - 16}{x - 2}$
5. $\dfrac{1 - a^5}{1 - a}$
6. $\dfrac{a^5 + 1}{a + 1}$
7. $\dfrac{a^6 - 64}{a + 2}$
8. $\dfrac{x^5 + 243}{x + 3}$

Also the following:

9. $m^4 - 16n^8$ by $m - 2n^2$. Ans. $m^3+2m^2n^2+4mn^4+8n^6$.
10. $32a^5+b^5$ by $2a+b$. Ans. $16a^4 - 8a^3b+4a^2b^2 - 2ab^3+b^4$.
11. $m^5 - 243n^5$ by $m - 3n$.
Ans. $m^4+3m^3n+9m^2n^2+27mn^3+81n^4$.

FACTORING.

43. Factoring is the process of resolving a quantity into its factors.

Monomials may be factored by inspection. For example:

$$72a^2b^3c^2 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3\ aabbbcc.$$

NOTE. A dot may be placed between the figures instead of the sign $\times$, the same operation (multiplication) being indicated.

EXAMPLES FOR PRACTICE.

Factor the following by inspection:

1. $12a^4b^3$.
2. $13a^2b$.
3. $8x^3y^3$
4. $10m^4b^6$.

A polynomial cannot always be factored, but there are certain forms which are always factorable. The following cases are the more important:

Case I. *When the terms of the polynomial have a common monomial factor.*

For example, factor $2ab + a^2 - 3a^2c$.

Each term contains the monomial factor a, i. e., a is *common* to all the terms. Dividing by a we have $2b + a - 3a^2c$. Hence we may write the following equation:

$$2ab + a^2 - 3a^2c = a\ (2b + a - 3a^2c).$$

Factor $x^3y^2 - x^2y$. Here x^2y is common to both terms, and

$$x^3y^2 - x^2y = x^2y\ (xy - 1),$$

That is $xy - 1$ multiplied by x^2y equals $x^3y^2 - x^2y$.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $x^2 + 5x$. Ans. $x\ (x + 5)$.
2. $16a^4 - 12a$. Ans. $4a\ (4a^3 - 3)$.

3. $27c^4d^2 + 9c^3d$. Ans. $9c^3d\ (3cd + 1)$.
4. $a^5 - 2a^4 - 3a^3 - a^2$. Ans. $a^2\ (a^3 - 2a^2 + 3a - 1)$.
5. $21m^3n + 35mn^3 - 14mn$. Ans. $7mn\ (3m^2 + 5n^2 - 2)$.

Case II. *When the polynomial consists of four terms, of which the first two and the second two contain a common binomial factor.*

Factor $ax - bx + ay - by$.

Factoring the first two and the last two terms by Case I, we have

$$x\ (a - b) + y\ (a - b).$$

Each term now contains the binomial factor $a - b$ and we divide by it just as we did in Case I with the common monomial factor. Hence dividing by this binomial factor $a - b$ we have $x + y$; that is

$$ax - bx + ay - by = x\ (a - b) + y\ (a - b) = (a - b)\ (x + y)$$

Factor $am + bm - an - 2bn$

$$\begin{aligned} am + 2bm - an - 2bn &= am + 2bn - (an + 2bn) \\ &= m\ (a + 2b) - n\ (a + 2b). \\ &= (a + 2b)\ (m - n). \end{aligned}$$

Note. If, as in the above, the third term is negative, it is well to enclose the last two terms in a parenthesis as above, the signs of the last two terms being the opposite in the parenthesis of what they were out of it.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $ac - cm + ad - dm$. Ans. $(a - m)\ (c + d)$.
2. $x^2 + 2x - xy - 2y$. Ans. $(x - y)\ (x + 2)$.
3. $a^3 - a^2b - ab^2 + b^3$. Ans. $(a^2 - b^2)\ (a - b)$.
4. $mx^2 - my^2 + nx^2 - ny^2$. Ans. $(m + n)\ (x^2 - y^2)$.
5. $6x^3 + 4x^2 - 9x - 6$. Ans. $(2x^2 - 3)\ 3x + 2)$.
6. $6n - 21m^2n - 8m + 28m^3$. Ans. $(3n - 4m)\ (2 - 7m^2)$.

Perfect squares. If a quantity can be resolved into two equal factors it is said to be a perfect square, each of the equal factors being the square root of the quantity. Thus 9 may be broken up into two equal factors of 3, for $3 \times 3 = 9$. Again $4a^2b^4$ is a perfect square, as it has $2ab^2$ for its square root.

From the preceding we derive the following rule for extracting the square root of a monomial.

Extract the square root of the coefficient, and divide the exponent of each letter by 2.

Find the square root of $64a^4b^6c^8$.

The square root of 64 is 8, and 4, 6, 8 each divided by 2 gives 2, 3, and 4 respectively. Hence our answer is $8a^2b^3c^4$.

From Article 41 we see that a trinomial is a perfect square when its first and last terms are perfect squares and are *positive*, and the second term is twice the product of their square roots. For example $9a^2 + 12ab + 4b^2$ is a perfect square, because $3a$ is a square root of $9a^2$, $2b$ is a square root of $4b^2$ and because $12ab$ is twice the product of $3a$ and $2b$, or in other form, $3a \times 2b \times 2 = 12ab$. This may be clearly seen by observing the following:

$$\begin{array}{r} 3a + 2b \\ 3a + 2b \\ \hline 9a^2 + 6ab \\ \quad 6ab + 4b^2 \\ \hline 9a^2 + 12ab + 4b^2 \end{array}$$

If we had $-12ab$ *either* $3a$ or $2b$ would have to be negative.

Hence, in order to obtain the square root or a perfect trinomial square, we take the converse of the rules of Article 41.

Extract the square roots of the first and last terms, and connect the results by the sign of the second term.

This brings us to Case III.

Case III. *When a trinomial is a perfect square.*

Factor $a^2 + 4ab^2 + 4b^4$.

$a^2 + 4ab^2 + 4b^4 = (a + 2b^2)\ (a + 2b^2) = (a + 2b^2)^2$.

Factor $4x^2 - 12xy + 9y^2$.

$4x^2 - 12xy + 9y^2 = (2x - 3y)\ (2x - 3y) = (2x - 3y)^2$.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $4 + 4m + m^2$. Ans. $(2 + m)^2$.
2. $a^2 - 10a + 25$. Ans. $(a - 5)^2$.
3. $x^4 + 12x^2 + 36$. Ans. $(x^2 + 6)^2$.
4. $a^4 + 2a^3 + a^2$. Ans. $(a^2 + a)^2$.

5. $25x^2 - 70xyz + 49y^2z^2$. Ans. $(5x - 7yz)^2$.
6. $4a^4b^2 + 52a^2b^3 + 169a^2b^4$. Ans. $2a^2b + 13ab^2)^2$.
7. $16x^4 - 120\,mnx^3 + 225m^2n^2$. Ans. $(4x^2 - 15mn)^2$.
8. $64x^8 - 160x^7 + 100x^6$. Ans. $(8x^4 - 10x^3)^2$.
9. $(x + y)^2 - 16\,(x + y) + 64$. Ans. $(x + y - 8)^2$.
10. $(x^2 - x)^2 + 6\,(x^2 - x) + 9$. Ans. $(x^2 - x + 3)^2$.

Case IV. *When an expression is the difference of two perfect squares.*

By comparison with the third case of Article 41, we see that such an expression as the above is the product of the sum and the difference of two quantities. Hence, to obtain the factors:

Extract the square root of the first term and of the last term, and take the sum of the square roots for one factor and their difference for the other factor, retaining the same order.

Factor $9a^2 - 16b^2$.

The square root of $9a^2$ is $3a$, that of $16b^2$ is $4b$. Their sum is $3a + 4b$, their difference, retaining the same order, is $3a - 4b$. Hence we have $9a^2 - 16b^2 = (3a + 4b)\,(3a - 4b)$.

If we have instead of perfect monomial squares, other quantities which are perfect squares, we proceed in the same way. For example factor:

$(x + 2y)^2 - (2x - y)^2$.

$(x + 2y)^2 - (2x - y)^2$

$= [(x + 2y) + (2x - y)]\,[(x + 2y) - (2x - y)]$

$= (x + 2y + 2x - y)\,(x + 2y - 2x + y)$.

$= (3x + y)\,(3y - x)$.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $x^2 - 4y^2$. Ans. $(x + 2y)\,(x - 2y)$.
2. $a^4b^6 - 9c^8$. Ans. $(a^2b^3 + 3c^4)\,(a^2b^3 - 3c^4)$.
3. $36y^2 - 169x^4$. Ans. $(6y + 13x^2)\,(6y - 13x^2)$.
4. $a^2b^4c^2 - 1$. Ans. $(ab^2c + 1)\,(ab^2c - 1)$.
5. $64 - m^4$. Ans. $(8 + m^2)\,(8 - m^2)$.
6. $1 - 121x^2$. Ans. $(1 + 11x)\,(1 - 11x)$.
7. $49a^4 - 36b^2$. Ans. $(7a^2 + 6b)\,(7a^2 - 6b)$.

8. $144y^2 - 1$. Ans. $(12y + 1)\ (12y - 1)$.
9. $9x^2 - 49y^2$. Ans. $(3x + 7y)\ (3x - 7y)$.
10. $16m^2n^2 - 256$. Ans. $(4mn + 16)\ (4mn - 16)$.

Case V. *When an expression is a trinomial of the form,* $x^2 + mx + n$.

We have derived a rule for the product of two trinomials. By this rule we saw that $x + a$ multiplied by $x + b$ equals $x^2 + (a + b)\ x + ab$. In this trinomial product we note that in the second term, the coefficient of x, $a + b$ is the *sum* of the second terms of the binomials $x + a$ and $x + b$, and that the last term of the trinomial ab, is the *product* of the second terms of the binomials. Hence if we have a trinomial of the form $x^2 + mx + n$ to resolve into two binomials, we place as the first term of each binomial x, and as the second terms two numbers whose sum is m and whose product is n.

NOTE. In the first part of this consideration, m was represented by $a + b$ and n by ab.

Let us take a few examples of the proceeding:

$$(x + 2)\ (x + 3) = x^2 + 5x + 6.$$
$$(x + 2)\ (x - 3) = x^2 - x - 6.$$
$$(x - 2)\ (x + 3) = x^2 + x - 6.$$
$$(x - 2)\ (x - 3) = x^2 - 5x + 6.$$
$$(x + 7)\ (x - 2) = x^2 + 5x - 14.$$

In the above we see that in each case the coefficient of x (considering signs) is the *sum* of the last terms of the binomials, and the third term is the product of the last terms of the binomials.

We may reverse the order of the proceeding:

$$x^2 + 5x + 6 = (x + 2)\ (x + 3)$$
$$x^2 - x - 6 = (x + 2)\ (x - 3) \text{ etc.}$$

Hence, to factor a trinomial such as the above:

Find two numbers whose algebraic sum is the coefficient of x *and whose product is the last term.* Make x in each case the first term of the factors and the two numbers thus found the second terms.

The work of finding the numbers may be lessened by observing the following:

1. When the sign of the last term is +, the coefficient of x is the numerical sum of both numbers, which are + if the second term is +, and — when the second term is —.

2. When the sign of the last term is —, the coefficient of x is the *numerical* difference between the two numbers which are opposite in sign, the greater of the two numbers having the same sign as the coefficient of x.

The above may be embodied in the two following rules.

I. *If the last term is +, find two numbers whose sum is the coefficient of x, and whose product is the last term. Give both numbers the sign of the second term.*

II. *If the last term is —, find two numbers whose difference is the coefficient of x, and whose product is the last term. Give the greater number the sign of the second term and the smaller number the opposite sign.*

By "coefficient of x" and "last term" in the preceding, the *absolute values* are designated with no regard to sign.

Let us take a few examples to illustrate the preceding work.

Factor $x^2 + 9x + 18$.

We must now find two numbers whose sum is 9 and whose product is 18.

These numbers are, obviously, 6 and 3.

Hence, as all signs are +,

$$x^2 + 9x + 18 = (x + 6)\ (x + 3).$$

Factor $x^2 - 7x + 10$.

Here we are to find two numbers whose sum is 7 and product 10.

The numbers are evidently 5 and 2.

The sign of the second term is —, and therefore both numbers are —.

Hence,

$$x^2 - 7x + 10 = (x - 5)\ (x - 2).$$

It is perhaps easier to say: Find two numbers whose sum is — 7 and product + 10. These numbers are, of course, — 5 and — 2, for $(-5) + (-2) = -5 - 2 = -7$ and (-5) times (-2) or $(-5)\ (-2) = +10$.

Factor $x^2 + 3x - 4$.

We must here find two numbers whose difference is 3 and whose product is 4. These numbers are 4 and 1. As the second term is +, the greater number is + and the lesser —.

Hence,

$$x^2 + 3x - 4 = (x + 4)\ (x - 1).$$

We could have, as in the previous case, proceeded more directly by taking two numbers whose sum was + 3 and product — 4. The numbers are + 4 and — 1, for $+ 4 - 1 = 3$ and $4\ (-1) = -4$.

Factor $x^2 - 7x - 30$.

We must now find two numbers whose difference is 7 and whose product is 30.

These numbers are 10 and 3.

Hence, as the second term is — making the greater number —, and the lesser +, we have

$$x^2 - 7x - 30 = (x - 10)\ (x + 3).$$

The more direct procedure is to find two numbers whose sum is — 7 and product — 30.

These are, of course, — 10 and + 3, for $-10 + 3 = -7$ and $(-10)\ (+3) = -30$.

EXAMPLES FOR PRACTICE.

Factor the following:

1.	$x^2 + 15x + 36$.	Ans. $(x + 12)\ (x + 3)$.
2.	$x^2 + 14x + 45$.	Ans. $(x + 9)\ (x + 5)$.
3.	$x^2 + 13x + 36$.	Ans. $(x + 9)\ (x + 4)$.
4.	$x^2 - 3x + 2$.	Ans. $(x - 2)\ (x - 1)$.
5.	$a^2 - 11a + 18$.	Ans. $(a - 9)\ (a - 2)$.
6.	$c^2 - 18c + 32$.	Ans. $(c - 16)\ (c - 2)$.
7.	$m^2 - 8m - 33$.	Ans. $(m - 11)\ (m + 3)$.
8.	$x^2 - 9x - 90$.	Ans. $(x - 15)\ (x + 6)$.
9.	$x^2 - 11x + 10$.	Ans. $(x - 10)\ (x - 1)$.
10.	$y^2 - 2y - 35$.	Ans. $(y - 7)\ (y + 5)$.
11.	$x^2 - 5x - 24$.	Ans. $(x - 8)\ (x + 3)$.
12.	$x^2 + 5x - 14$.	Ans. $(x + 7)\ (x - 2)$.
13.	$m^2 + 16m + 63$.	Ans. $(m + 9)\ (m + 7)$.
14.	$m^2 + 38m - 80$.	Ans. $(m + 40)\ (m - 2)$.
15.	$x^2 + 4x - 96$.	Ans. $(x + 12)\ (x - 8)$.

16. $x^2 + 7x - 98$. Ans. $(x + 14)\ (x - 7)$.
17. $a^4 + 10a^2 - 144$. Ans. $(a^2 + 18)\ (a^2 - 8)$.
18. $x^2y^6 + 2xy^3 - 120$. Ans. $(xy^3 + 12)\ (xy^3 - 10)$.
19. $y^8 - 20y^4 + 91$. Ans. $(y^4 - 13)\ (y^4 - 7)$.

It is not a requirement that letters should not be contained in the last term, or that all three terms should be monomials.

The same rules are observed in the following examples:

20. $x^4 + 3abx^2 - 108a^2b^2$. Ans. $(x^2 + 12ab)\ (x^2 - 9ab)$.
21. $1 + 5mn^2 - 84m^2n^4$. Ans. $(1 + 12mn^2)\ (1 - 7mn^2)$.
22. $(m - n)^2 - 3\ (m - n) - 4$.

Ans. $(m - n - 4)\ (m - n + 1)$.

If the quantity could be resolved into three *equal* factors, it is called a *perfect cube*, and each equal factor is its *cube root*.

Since $512x^3y^6$ equals $8xy^2 \times 8xy^2 \times 8xy^2$, it is a perfect cube, and $8xy^2$ is its cube root.

Hence, we may find the cube root of a monomial by *extracting the cube root of the coefficient, and prefixing it to the letters contained, the exponents of those letters being first divided by* 3.

The cube root of $343a^6b^3c^9$ is $7a^2bc^3$.

Case VI. *When an expression is the sum or difference of two perfect cubes.*

If we divide $a^3 + b^3$ by $a + b$, we have as a result $a^2 - ab + b^2$. To write it differently:

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$$

If we divide $a^3 - b^3$ by $a - b$ we have $a^2 + ab + b^2$. Written differently:

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$

Hence, if we wish to factor the *sum* of two perfect cubes, *we write as the first factor the sum of their cube roots and as the other factor the square of the cube root of the first, minus the product of the cube roots of the two, plus the square of the cube root of the second.*

In the case of the *difference* of two perfect cubes we proceed in the same way *except that as the first factor we write the cube root of the first minus the cube root of the second, and that in the second factor all signs are positive.*

Factor $8x^3 + 27y^6$.

The cube root of the first is $2x$ and that of the second $3y^2$.

Hence we have

$8x^3 + 27y^6 = (2x + 3y^2)\ (4x^2 - 6xy^2 + 9y^4)$.

Factor $125m^3 - 343n^6$.

The cube root of the first is $5m$ and that of the second $7n^2$.

Hence we have

$125m^3 - 343n^6 = (5m - 7n^2)\ (25m^2 + 35mn^2 + 49n^4)$.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $a^3 + x^3$. Ans. $(a + x)\ (a^2 - ax + x^2)$.
2. $x^{12} + y^6$. Ans. $(x^4 + y^2)\ (x^8 - x^4y^2 + y^4)$.
3. $a^2b^2 - c^2$. Ans. $(ab - c)\ (a^2b^2 + abc + c^2)$.
4. $8m^3 - 1$. Ans. $(2m - 1)\ (4m^2 + 2m + 1)$.
5. $m^6n^3 - a^3$. Ans. $(m^2n - a)\ (m^4n^2 + am^2n + a^2)$.
6. $27a^6b^3 + 64c^6$.

   Ans. $(3a^2b + 4c^2)\ (9a^6b^2 - 12a^2bc^2 + 16c^4)$.

The second factor in all the above examples is not factorable. This applies also to the examples of the following case.

Case VII. *When an expression is the sum or difference of two equal odd powers of two quantities.*

When an expression is the sum or difference of two equal odd powers of two quantities, it is divisible by the sum or difference respectively of those quantities, and may, therefore, be factored by inspection in the following manner, according to the method previously shown.

Suppose it is required to factor $a^5 + b^5$.

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Therefore we have

$a^5 + b^5 = (a + b)\ (a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

In the same way we find that

$a^5 - b^5 = (a - b)\ (a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

EXAMPLES FOR PRACTICE.

Factor the following:

1. $x^5 + 1$. Ans. $(x + 1)(x^4 - x^3 + x^2 - x + 1)$.
2. $1 - a^5$. Ans. $(1 - a)(1 + a + a^2 + a^3 + a^4)$.
3. $m^7 + n^7$.

Ans. $(m + n)(m^6 - m^5n + m^4n^2 - m^3n^3 + m^2n^4 - mn^5 + n^6)$.

4. $m^7 - n^7$.

Ans. $(m - n)(m^6 + m^5n + m^4n^2 + m^3n^3 + m^2n^4 + mn^5 + n^6)$.

5. $a^7 - 1$.

Ans. $(a - 1)(a^6 + a^5 + a^4 + a^3 + a^2 + a + 1)$.

6. $a^5 - b^5c^5$.

Ans. $(a - bc)(a^4 + a^3bc + a^2b^2c^2 + ab^3c^3 + b^4c^4)$.

7. $1 + 32m^5$.

Ans. $(1 + 2m)(1 - 2m + 4m^2 - 8m^3 + 16m^4)$.

8. $a^7 + 128$.

Ans. $(a + 2)(a^6 - 2a^5 + 4a^4 - 8a^3 + 16a^2 - 32a + 64)$.

9. $243x^5 - 32$.

Ans. $(3x - 2)(81x^4 + 54x^3 + 36x^2 + 24x + 16)$.

Expressions may, by applying one or more of the preceding rules, be separated into more than two factors. Always be careful to remove monomial factors whenever possible.

Factor $2ax^2 - 18ay^2$.

By Case I, $2ax^2 - 18ay^2 = 2a(x^2 - 9y^2)$.

As the quantity in the parenthesis may be factored by Case IV., we have the following result:

$2ax^2 - 18ay^2 = 2a(x + 3y)(x - 3y)$.

Factor $a^2m - b^2m + a^2n - b^2n$.

By Case II, $a^2m - b^2m + a^2n - b^2n = (a^2 - b^2)(m + n)$.

Factoring $a^2 - b^2$ by Case IV, we have

$a^2m - b^2m + a^2n - b^2n = (a + b)(a - b)(m + n)$.

Factor $x^6 - y^6$.

By Case IV, $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$.

By Case VI, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Hence,

$(x^6 - y^6) = (x + y)(x - y)(x^2 - xy + y^2)(x^2 + xy + y^2)$.

Factor $m^8 - n^8$.

By Case IV, $m^8 - n^8 = (m^4 + n^4)\ (m^4 - n^4)$.

$= (m^4 + n^4)\ (m^2 + n^2)\ (m^2 - n^2)$.

$= (m^4 + n^4)\ (m^2 + n^2)\ (m + n)\ (m - n)$.

EXAMPLES FOR PRACTICE.

Factor the following by one or more cases as required:

1. $a + 4ax + 4ax^2$. Ans. $a(1 + 2x)^2$.
2. $m^6 - 1$.

Ans. $(m + 1)\ (m - 1)\ (m^2 - m + 1)\ (m^2 + m + 1)$.

3. $x^2 - (2y - 3z)^2$. Ans. $(x + 2y - 3z)\ (x - 2y + 3z)$.
4. $x^5 - x^4 + x^3 - x^2$. Ans. $x^2(x^2 + 1)\ (x - 1)$.
5. $m^2 + 20mn + 100n^2$. Ans. $(m + 10n)^2$.
6. $a^5b^5 - c^5d^5$.

Ans. $(ab - cd)\ (a^4b^4 + a^3b^3cd + a^2b^2c^2d^2 + abc^3d^3 + c^4d^4)$.

7. $4x^3 - x$. Ans. $x(2x^2 + 1)\ (2x^2 - 1)$.
8. $m^4 - 16$. Ans. $(m^2 + 4)\ (m + 2)\ (m - 2)$.
9. $5x^3 - 5$. Ans. $5(x - 1)\ (x^2 + x + 1)$.
10. $2x^3y + 2xy^3 - 2xyz^2 + 4x^2y^2$.

Ans. $2xy(x + y + z)\ (x + y - z)$.

11. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

Ans. $(x + y + z)\ (x + y - z)\ (x - y + z)\ (x - y - z)$.

12. $a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2$.

Ans. $(a + b)\ (a - b)\ (x + y)\ (x - y)$.

13. $(a^2 + 3a)^2 - 14(a^2 + 3a) + 40$.

Ans. $(a - 1)\ (a - 2)\ (a + 4)\ (a + 5)$.

14. $x^3 - x^2 - x + 1$. Ans. $(x - 1)^2\ (x + 1)$.
15. $x^3 - y^3 - 3xy\ (x - y)$. Ans. $(x - y)^3$.

HIGHEST COMMON FACTOR.

44. A **common factor** of two or more quantities is a quantity by which each of them may be divided without leaving a remainder.

Thus $2x^2y$ is a common factor of $8x^2y^3$ and $12x^3y^4$ as it is contained in the former $4y$ times and in the latter $6xy^2$ times.

A *prime quantity* is one which cannot be divided by any integral quantity other than itself and unity without leaving a remainder. For example x, y, a, b, $a + b$, etc. are prime quantities.

Two quantities are *prime to each other* when they have no common factor except unity.

Thus $3a$ and $2b^2$ are prime to each other.

The **highest common factor** of two or more quantities is the product of *all the prime factors common to those quantities.*

From this definition, the highest common factor of two or more quantities is evidently the expression of *highest degree* which will divide each of them without leaving a remainder.

Thus the highest common factor of a^2b^3, a^4b^4, *and* $a^3\ b^2$ *is* a^2b^2.

In the determination of highest common factor it is convenient to classify into three cases.

Case I. *When the quantities are monomials.*

Find the H. C. F. of $15a^2x$, $45a^3y^2$, and $60a^4z^3$.

$$15a^2x = 3 \times 5 \times a^2x.$$
$$45a^3y^2 = 3 \times 3 \times 5 \times a^3y^2.$$
$$60a^4z^3 = 2 \times 2 \times 3 \times 5 \times a^4z^3.$$

Hence the H. C. F, $= 3 \times 5 \times a^2 = 15a^2$, for 3 and 5 and a^2 are all common to each of the quantities of which the common factor is required. It is customary to place dots in place of the multiplication sign, in the following manner:

Find the H. C. F. of $36a^2b^3c$ and $42ab^2c$.

$$36a^2b^3c = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a^2b^3c.$$
$$42ab^2c = 2 \cdot 3 \cdot 7 \cdot ab^2c.$$

Hence H. C. F. $= 2 \cdot 3 \cdot ab^2c$.

Thus to find the H. C. F. we have the following rule:

To the highest common factors of the coefficients annex the common letters, giving to each the lowest exponent with which it occurs in any of the given quantities.

EXAMPLES FOR PRACTICE.

Find the H. C. F. of the following:

1. $42x^2y^2$, $70x^2yz$, and $98x^4y^3z$. Ans. $14x^2y$.
2. $169a^2bcd^4$, $338ab^2d^2$, $13a^4b$. Ans. $13ab$.

Case II. *When the quantities are polynomials which may be readily factored by inspection.*

Find the H. C. F. of

$a^2x^4 - a^2y^4$ and $ax^4 - 2ax^2y^2 + ay^4$.

By factoring

$$a^2x^4 - a^2y^4 = a^2 (x^2 + y^2) (x + y) (x - y).$$

$$ax^4 - 2ax^2y^2 + ay^4 = a (x + y)^2 (x - y)^2.$$

Hence H. C. F. $= a (x + y) (x - y)$.

EXAMPLES FOR PRACTICE.

Find the H. C. F. of

1. $a^2 + b^2$ and $a^4 - b^4$. Ans. $a + b$.
2. $x^2 + 2xy + y^2$ and $x^4y + xy^4$. Ans. $x + y$.
3. $am - an$, $a^2m^2 - a^2n^2$, $am^2 - 2amn + an^2$. Ans. $a (m - n)$.
4. $x^2 + 2x + 1$, $x^2 - 1$, and $x^3 + 1$. Ans. $x + 1$.
5. $m^3 - 8n^3$ and $m^2 + 6mn - 16n^2$. Ans. $m - 2n$.
6. $3a^2 + 12ab + 12b^2$ and $6a^3 + 48b^3$. Ans. $3 (a + 2b)$.
7. $4a^2 - 4a + 1$, $4a^2 - 1$, $2ax - x - 2ay + y$. Ans. $2a - 1$.

Case III. *When the quantities are polynomials which cannot be readily factored by inspection.*

In arithmetic we may determine the H. C. F. of two numbers by the following procedure:

Divide the greater number by the lesser. If there is a remainder, divide the divisor by it. Repeat this operation until there is no remainder. The last divisor is the H. C. F. required.

The following example illustrates this method:

Find the H. C. F. of 132 and 451.

$$\begin{array}{r} 132)451(3 \\ 396 \\ \hline 55)132(2 \\ 110 \\ \hline 22)55(2 \\ 44 \\ \hline 11)22(2 \\ 22 \\ \hline \end{array}$$

Hence 11, being the last divisor, is the H. C. F. which is required.

We may apply the same rule to algebraic expressions. The method is readily susceptible of proof.

Suppose that we call the two quantities A and B, the degree of A being not lower than that of B. To determine their H. C. F.:

Divide A by B. If there is a remainder divide B by that remainder, continue to make the divisor the dividend for each remainder left after dividing by that divisor, until there is no remainder. The last divisor is then the H. C. F. required.

In this work the following observation should be kept in mind:

1. Each division should be continued until the remainder is of a lower degree than the divisor.

2. The work should be kept in the same order of powers of some common letter.

3. Either given expression may be divided by any quantity not a factor of the other, for such a quantity evidently cannot form a part of the H. C. F. Also, similarly, any remainder may be divided by any quantity not a common factor of the given expressions.

4. If the first term of any remainder be negative the sign of each term may be changed.

5. If the first term of the dividend or of any remainder is not divisible by the first term of the divisor, it may be made divisible by multiplying the dividend or remainder by any quantity not a factor of the divisor.

6. If the quantities given have a common factor, which can be seen on inspection, remove it and ascertain the H. C. F. of the resulting expressions. The latter multiplied by the former is the H. C. F. of the quantities.

Let us take a few illustrations.

Find the H. C. F. of

$2x^2 + x - 6$ and $4x^2 - 8x + 3$.

$$\begin{array}{r} 2x^2 + x - 6 \,)\, 4x^2 - 8x + 3 \,(\, 2 \\ 4x^2 + 2x - 12 \\ \hline -10x + 15 \end{array}$$

Dividing $-10x + 15$ by -5 we have $2x - 3$.

$$2x - 3)2x^2 + x - 6(x + 2$$
$$2x^2 - 3x$$
$$\overline{\qquad\qquad}$$
$$4x - 6$$
$$4x - 6$$
$$\overline{\qquad\qquad}$$

Hence $2x - 3$ is the H. C. F.

Find the H. C. F. of

$2a^4 - 7a^3 + 5a^2 - 6a$ and $3a^3 - 7a^2 - 7a + 3$.

Remove from the former the common monomial factor of a and divide the latter by it, first multiplying the latter by 2, as 3 is not divisible by 2.

$$2a^3 - 7a^2 + 5a - 6)6a^3 - 14a^2 - 14a + 6(3$$
$$6a^3 - 21a^2 + 15a - 18$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$7a^2 - 29a + 24$$

As $2a^3$ is not divisible by $7a^2$ we multiply *each* term of the expression by 7.

$$7a^2 - 29a + 24)14a^3 - 49a^2 + 35a - 42(2a$$
$$14a^3 - 58a^2 + 48a$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$9a^2 - 13a - 42$$

To make the first term of this result of subtraction divisible by $7a^2$ we multiply it by 7.

$$7a^2 - 29a + 24)63a^2 - 91a - 294(9$$
$$63a^2 - 261a + 216$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$170a - 510$$

Dividing this remainder by 170 we have $a - 3$.

$$a - 3)7a^2 - 29a + 24(7a - 8$$
$$7a^2 - 21a$$
$$\overline{\qquad\qquad}$$
$$-8a + 24$$
$$-8a + 24$$
$$\overline{\qquad\qquad}$$

Hence $a - 3$ is the H. C. F.

EXAMPLES FOR PRACTICE.

Find the H. C. F. of

1. $2x^2 - 5x + 3$ and $4x^3 - 2x^2 - 9x + 7$. Ans. $x - 1$.
2. $a^3 - 8$ and $a^3 - 6a^2 + 11a - 6$. Ans. $a - 2$.

3. $6m^3 - m^2n - 5mn^2$ and $21m^3 - 26m^2n + 5mn^2$.

Ans. $m\ (m - n)$ or $m^2 - mn$.

To find the H. C. F. of three or more quantities, find the H. C. F. of two, then of this H. C. F. of two and another quantity, and so on. The last divisor will be the H. C. F. of all the quantities.

LOWEST COMMON MULTIPLE.

45. A **common multiple** of two or more quantities is a quantity susceptible of being divided by every one of those quantities without a remainder. Thus it follows that a common multiple of several quantities must contain all the prime factors of every one of those several quantities.

The **lowest common multiple** of two or more quantities is the product of their different prime factors, each of these prime factors being taken the greatest number of times which it occurs in any one of those quantities.

Obviously, from this definition, the lowest common multiple of two or more quantities is the expression of lowest degree which can be divided by each of them without a remainder.

For example, the lowest common multiple of x^3y, x^7y^3, and y^2z is x^7y^3z.

When quantities are prime to each other, their product is their lowest common multiple.

In the determination of the lowest common multiple of algebraic expressions, it is convenient to classify into the three following cases:

Case I. *When the quantities are monomials.*

Find the L. C. M. of $60a^2y^2$, $72a^3x$, and $84bx^2$.

$$60a^2y^2 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a^2y^2$$
$$72a^3x = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot a^3x$$
$$84bx^2 = 2 \cdot 2 \cdot 3 \cdot 7 \cdot bx^2$$

Hence, the L. C. M. $= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot a^3bx^2y^2$

$= 2520a^3bx^2y^2$.

EXAMPLES FOR PRACTICE.

Find the L. C. M. in the following:

(1.) $5a^5b^3$ and $15a^3b$. Ans. $15a^5b^3$.

(2.) $13x^5y$ and $169xz^2$. Ans. $169x^5yz^2$.

(3.) $30a^2x$ and $27abx^4$. Ans. $270a^2bx^4$.

(4.) $4m^2$, $9mn^3$, and $18m^2n$. Ans. $36m^2n^3$.

The following rule may prove helpful:

To the lowest common multiple of the coefficients annex all the letters which appear in the given quantities, giving each letter the highest exponent which it has in any of the given quantities.

Case II. *When the quantities are polynomials which can be readily factored by inspection.*

Find the L. C. M. of

$x^2 - x - 6$, $x^2 - 9$, and $x^3 - 6x^2 + 9x$.

$$x^2 - x - 6 = (x + 2)(x - 3).$$
$$x^2 - 9 = (x + 3)(x - 3).$$
$$x^3 - 6x^2 + 9x = x(x + 3)^2.$$

Hence the L. C. M. $= x(x + 3)^2(x + 2)(x - 3)$.

EXAMPLES FOR PRACTICE.

Find the L. C. M. in the following:

1. $a^2 - b^2$ and $ac + bc$. Ans. $c(a^2 - b^2)$.
2. $ax + ay$ and $bx^2 + 2bxy + by^2$. Ans. $ab(x + y)^2$.
3. $m^3 + n^3$ and $m^2 - n^2$. Ans. $(m^2 - n^2)(m^2 - mn + n^2)$.

Case III. *When the quantities are polynomials which cannot be readily factored by inspection.*

Let A and B represent two expressions and let F be their highest common factor. Suppose that F is contained a times in A and b times in B. That is $A = aF$ and $B = bF$, then

$$A \times B = abF^2. \qquad (1)$$

As a and b can have no common factor, the L. C. M. of aF and bF is abF. Letting M represent the L. C. M. of A and B (i.e., aF and bF) we have $M = abF$.

Multiply M by F we have $M \times F$, and multiplying abF by F we have abF^2.

Therefore,

$$F \times M = abF^2 \qquad (2)$$

Now $A \times B$ and $F \times M$ are both equal to abF^2. Therefore,

$$A \times B = F \times M \qquad (3)$$

or
$$\frac{A \times B}{F} = M$$

For, by (3), we have, that the *product of any two quantities is equal to the product of their highest common factor and lowest common multiple.*

Therefore, to find the L. C. M. of two quantities:

Divide their product by their highest common factor, or divide one of the quantities by the highest common factor of the two, and multiply the other quantity by the quotient thus obtained.

Find the L. C. M. of

$2x^2 + x - 6$ and $4x^2 - 8x + 3$.

By Article 44, their highest common factor is $2x - 3$ and $2x^2 + x - 6$ divided by $2x - 3$ is $x + 2$, and $4x^2 - 8x + 3$ multiplied by $x + 2$ is

$$\begin{array}{r} 4x^2 - 8x + 3 \\ x + 2 \\ \hline 4x^3 - 8x^2 + 3x \\ 8x^2 - 16x + 6 \\ \hline 4x^3 - 13x + 6 \end{array}$$

Hence L. C. M. $= 4x^3 - 13x + 6$.

EXAMPLES FOR PRACTICE.

Find the L. C. M. of

$8a^2 + 30a + 7$ and $12a^2 - 29a - 8$.

Ans. $24a^3 + 26a^2 - 219a - 56$.

FRACTIONS.

46. In arithmetic $\frac{2}{3}$ signifies 2 *divided by* 3. Similarly the expression $\frac{a}{b}$ signifies a divided by b. In other words, a units are divided into b parts and one of these equal parts taken, or *one* unit is divided into b equal parts and a parts taken.

Such an expression as $\frac{a}{b}$ is termed a **fraction.** In such a fraction the part above the line (a) is called the *numerator*, as in

arithmetic, and the part below the line (b) the *denominator*. The numerator and denominator are called the *terms* of the fraction.

The line between them indicates division just as does the sign $\div$. Thus $\frac{a}{b}$ is the same as $a \div b$.

The denominator shows the number of parts into which the unit is divided, and the numerator shows the number of parts taken.

An **entire quantity** or **integer** is one which has no fractional part, such as $a + b$, or $a + 3xy$, etc. Every integer may be considered as a fraction, the denominator of which is unity; thus $\frac{a+b}{1}$ and $\frac{a+3xy}{1}$ are integers, and are identical with integral expressions $a + b$ and $a + 3xy$.

We have already seen what a fractional quantity is.

A mixed quantity is one which has both entire and fractional parts, such as $a + \frac{x}{y}$, $x + \frac{a}{b}$, etc.

It is very important to understand the following general principles governing fractions:

(1.) *If the numerator of a fraction be multiplied or the denominator of a fraction divided by any quantity, the fraction is multiplied by that quantity.*

If we multiply the numerator of the fraction $\frac{a}{b}$ by c, we have $\frac{ac}{b}$. In the first fraction we have the unit divided into b parts and a parts taken. In the second case we have ac parts taken. Hence since c times as many parts were taken in the second case, $\frac{ac}{b} = \frac{a}{b} \times c$, i. e., multiplying the numerator of a fraction by a quantity multiplies the fraction by that quantity.

Again let $\frac{a}{b}$ be any fraction. Let the denominator b be divided by c. Then in the first case we have a parts of a unit divided into b parts, while in the second case we have a parts of a unit divided into $\frac{b}{c}$ parts. Consequently in the second case the number of parts will be only $\frac{1}{c}$ times as many as before, and each part will, therefore, be c times as great, or, as each part is c times as great, the whole number of parts is c times as great as before, hence, $\frac{a}{b \div c} = \frac{ac}{b}$.

Hence dividing the denominator of a fraction by a quantity is equivalent to multiplying the numerator by that quantity.

In a similar way we may prove that:

(2). *If the numerator of a fraction be divided, or the denominator multiplied by any quantity, the fraction is divided by that quantity.*

This is also evident as the result of such an operation would, obviously, have the opposite effect to that of the principle previously demonstrated in this article.

(3). *If the numerator and denominator of a fraction are both multiplied, or both divided, by the same quantity, the value of the fraction is not altered.*

For by (1) and (2) multiplying the numerator multiplies the fraction while multiplying the denominator divides it, and dividing the numerator divides the fraction while dividing the denominator multiplies it. Hence, in the first place we multiply and divide and in the second case divide and multiply the fraction by the same quantity, which obviously does not alter its value.

REDUCTION OF FRACTIONS TO LOWEST TERMS.

47. A fraction is in its **lowest terms** when its numerator and denominator are *prime to each other.*

It is convenient to classify into two cases.

Case I. *When the numerator and denominator can be readily factored by inspection.*

Since dividing both numerator and denominator by the same quantity (which is equivalent to cancelling equal factors in each of them) does not change the value of the fraction, we may proceed by the following rule:

Resolve both numerator and denominator into their prime factors, and cancel all of these factors which are common to both.

Let us take a few examples:

Reduce $\frac{10a^2b^7c}{15a^4b^5c^3}$ to its lowest terms:

$$\frac{10a^2b^7c}{15a^4b^5c^3} = \frac{2 \times 5a^2b^7c}{3 \times 5a^4b^5c^3}$$

Dividing both terms by $5a^2b^5c$, we have

$$\frac{2b^2}{3a^2c}$$

Reduce $\dfrac{3x^2 - 3y^2}{3x^2 - 6xy - 9y^2}$

$$\frac{3x^2 - 3y^2}{3x^2 - 6xy - 9y^2} = \frac{3(x+y)\ (x-y)}{3(x-3y)\ (x+y)}$$

$$= \frac{x-y}{x-3y}$$

If all factors of a term are removed by cancellation, unity, 1, which is a factor of all algebraic expressions, remains in place of the cancelled term.

If this occurs in the denominator, we have a case of exact division, and, as the denominator becomes 1, we have an entire quantity or integer as a result.

EXAMPLES FOR PRACTICE.

Reduce the following to lowest terms:

1. $\dfrac{2a^2b^4c}{5a^3bc^3}$ Ans. $\dfrac{2b^3}{5ac^2}$

2. $\dfrac{32mn}{64m^4n^3}$ Ans. $\dfrac{1}{2m^3n^2}$

3. $\dfrac{115c^3x^2y}{23c^2x^2}$ Ans. $5cy$.

4. $\dfrac{2a^2cd + 2abcd}{6a^2xy + 6abxy}$ Ans. $\dfrac{cd}{3xy}$

5. $\dfrac{x^2 - 2x - 15}{x^2 + 10x + 21}$ Ans. $\dfrac{x-5}{x+7}$

6. $\dfrac{4m^2 - 20m + 25}{4m^2 - 25m}$ Ans. $\dfrac{2m-5}{m(2m+5)}$

7. $\dfrac{x^2 - 4y^2}{x^2 + xy - 6y^2}$ Ans. $\dfrac{x+2y}{x+3y}$

8. $\dfrac{8a^3 + b^3}{4a^3 - 2a^2b + ab^2}$ Ans. $\dfrac{2a+b}{a}$

9. $\dfrac{m^3 - m^2 + 2m - 2}{2m^3 + m^2 + 4m + 2}$ Ans. $\dfrac{m-1}{2m+1}$

10. $\dfrac{x^2 - (y+z)^2}{(x-y)^2 - z^2}$ Ans. $\dfrac{x+y+z}{x-y+z}$

Case II. ***When the numerator and denominator cannot be readily factored by inspection.***

Since the highest common factor of two quantities is the product of their common prime factors, we have the following rule:

Divide both terms by the highest common factor of those terms.

Let us take an example:

Reduce $\frac{2x^2 + x - 6}{4x^2 - 8x + 3}$ to its lowest terms.

By Case III of H. C. F., the H. C. F. of both terms is $2x - 3$.

Hence dividing both numerator and denominator by $2x - 3$ we have:

$$\frac{2x^2 + x - 6}{4x^2 - 8x + 3} = \frac{x + 2}{2x - 1}$$

EXAMPLES FOR PRACTICE.

1. $\frac{a^2 - 6a + 5}{3a^2 + 4a - 7}$ Ans. $\frac{a - 5}{3a + 7}$

2. $\frac{2x^2 - 5x + 3}{12x^2 - 28x + 15}$ Ans. $\frac{x - 1}{6x - 5}$

If both terms of a fraction are multiplied by the same quantity the value of the fraction is not altered. Thus we may multiply both terms of a fraction by -1 without changing its value. Thus $\frac{a}{b} = \frac{-a}{-b}$

This may also be understood by remembering that, as a fraction represents the quotient obtained by dividing its numerator by its denominator, *if the signs of both terms are the same the fraction is positive*, but *if they are unlike in sign the fraction is negative.*

Hence if $\frac{a}{b} = x$, $\frac{-a}{-b} = x$ and $-\frac{a}{-b} = x$, and $-\frac{-a}{b} = x$ (for two negative signs result in a + sign when multiplied.)

But $\frac{-a}{b} = -x$ and $\frac{a}{-b} = -x$ just as $-\frac{a}{b} = -x$.

If either the numerator or the denominator is a polynomial and its sign is to be changed, *the sign of each of its terms* must be changed. If in the fraction $\frac{a-b}{c-d}$ the signs of both terms are changed we have the following:

$$\frac{a-b}{c-d} = \frac{-(a-b)}{-(c-d)} = \frac{-a+b}{-c+d} \text{ or } \frac{b-a}{d-c}$$

As the multiplication of quantities with like signs gives +, and the multiplication of quantities with unlike signs gives —, we may write the fraction $\frac{ab}{cd}$ in any of the following forms:

$$\frac{(-a)(-b)}{cd}, \frac{ab}{(-c)(-d)}, \frac{(-a)(-b)}{(-c)(-d)}, \frac{(-a)b}{(-c)d},$$

$$\frac{(-a)b}{c(-d)}, \text{ etc., or } -\frac{(-a)b}{cd}, -\frac{ab}{(-c)d}, -\frac{(-a)b}{(-c)(-d)}, \text{ etc.}$$

From this we see that,

If the terms of a fraction are composed of factors, changing the signs of any even number of factors does not change the value of the fraction. But changing the signs of an odd number of factors changes the sign of the fraction.

The fraction $\frac{a-b}{(c-d)(e-f)}$ may, therefore, be written in any of the following forms:

$$\frac{a-b}{(d-c)(f-e)}, \frac{b-a}{(d-c)(e-f)}, -\frac{b-a}{(d-c)(f-e)}$$

REDUCTION OF FRACTIONS TO ENTIRE OR MIXED QUANTITIES.

Since a fraction expresses division, we have the following rule:

Divide the numerator by the denominator.

Reduce $\frac{6x^2+4x-3}{2x}$ to a mixed quantity.

Dividing each term of the numerator by the denominator we have:

$$\frac{6x^2}{2x} + \frac{4x}{2x} - \frac{3}{2x} = 3x + 2 - \frac{3}{2x}$$

Reduce $\frac{x^3 - y^3}{x^2 + xy + y^2}$ to a whole or mixed quantity.

Dividing $x^3 - y^3$ by $x^2 + xy + y^2$, we have $x - y$ as a result. The answer is, therefore, $x - y$.

Reduce $\frac{x^3 - x^2 - x - 2}{x^2 + x - 1}$ to a mixed quantity.

Dividing we obtain $x - 2$ as a result, and a remainder of $2x - 4$. Hence our answer is $x - 2 + \frac{2x - 4}{x^2 + x - 1}$

NOTE. When the first term of the remainder will not contain the first term of the divisor, the former may be written over the latter in the form of a fraction.

If we have a fractional form like the following: $\frac{-a + b}{a + b + c}$ it should be written thus: $-\frac{a - b}{a + b + c}$

EXAMPLES FOR PRACTICE.

Reduce the following to mixed quantities:

1. $\frac{6a^3 - 3a^2 + 9a - 7}{3a}$ Ans. $2a^2 - a + 3 - \frac{7}{3a}$
2. $\frac{12m^2 - 8m - 7}{4m - 1}$ Ans. $3m - \frac{5m + 7}{4m - 1}$
3. $\frac{2x^4 - x^3 - 9x^2 + 14}{2x^2 - x - 3}$ Ans. $x^2 - 3 - \frac{3x - 5}{2x^2 - x - 3}$

REDUCTION OF MIXED QUANTITIES TO FRACTIONAL FORMS.

48. As this operation is the converse of the preceding, we have the following rule:

Multiply the integral part by the denominator of the fraction. Add or subtract, as the sign may indicate, the numerator of the fraction to this product. Write the expression over the denominator of the fraction.

Reduce $x^2 + x - \frac{4}{7}$ to fractional form.

Multiplying $x^2 + x$ by 7 we have $7x^2 + 7x$. Adding the numerator of the fractional part we have $7x^2 + 7x - 4$. Writing this over the denominator we obtain the answer $\frac{7x^2 + 7x - 4.}{7}$

Reduce $\frac{m-5}{2m-3} + m - 2$ to a fractional form.

$$\frac{m-5}{2m-3} + m - 2 = \frac{m - 5 + (m-2)(2m-3)}{2m-3}$$
$$= \frac{m - 5 + 2m^2 - 7m + 6}{2m-3}$$
$$= \frac{2m^2 - 6m + 1}{2m-3}$$

NOTE. It should be remembered that the line in a fraction acts as a vinculum. It is well, therefore, to enclose the numerator in a parenthesis when the sign before a fraction is —.

EXAMPLES FOR PRACTICE.

Reduce the following to fractional form:

1. $a + 1 - \frac{4}{a+3}$ Ans. $\frac{a^2 + 4a - 1}{a+3}$
2. $x + y - \frac{x^2 + y^2}{x+y}$ Ans. $\frac{2xy}{x+y}$
3. $7m - 3 - \frac{53m - 20}{8}$ Ans. $\frac{3m-4}{8}$

REDUCTION OF FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR.

49. Reduce $\frac{6a^2b}{17c^4d}$, $\frac{m^4n^2}{c^2d^2}$, and $\frac{5xy^2}{51c^3d^4}$ to equivalent fractions having the lowest common denominator.

The lowest common denominator is the lowest common multiple of the denominators $17c^4d$, c^2d^2, and $51c^3d^4$. It is found to be $51c^4d^4$.

As we have seen, both terms of a fraction may be multiplied by the same quantity without changing its value.

Multiplying both terms of $\frac{6a^2b}{17c^4d}$ by $3d^3$ we have $\frac{18a^2bd^3}{51c^4d^4}$

Multiplying both terms of $\frac{m^4n^2}{c^2d^2}$ by $51c^2$ we have $\frac{51c^2m^4n^2}{51c^4d^4}$

Multiplying both terms of $\frac{5xy}{51c^3d^4}$ by cd we have $\frac{5cdxy}{51c^4d^4}$

Hence the fractions required are:

$$\frac{18a^2bd^3}{51c^4d^4}, \quad \frac{51c^2m^4n^2}{51c^4d^4}, \quad \text{and} \quad \frac{5cdxy}{51c^4d^4}$$

Inspection of the above shows that the terms of each fraction are multiplied by a quantity obtained by dividing the lowest common denominator by its own denominator. We may, therefore, formulate the following rule:

Find the lowest common denominator. Divide this by each denominator separately. Multiply the corresponding numerators by the quotients, and write the results over the common denominator.

NOTE. Each fraction should be in its lowest terms to start with.

EXAMPLES FOR PRACTICE.

Reduce the following to equivalent fractions having the lowest common denominator:

1. $\frac{2}{a^2b^2}$, $\frac{3}{ab^3}$, and $\frac{4}{a^2b}$ Ans. $\frac{2b}{a^2b^3}$, $\frac{3a^2}{a^2b^3}$, and $\frac{4ab^2}{a^2b^3}$

2. $\frac{1}{m^3-1}$ and $\frac{1}{m^2-1}$

Ans. $\frac{m+1}{(m+1)(m^3-1)}$ and $\frac{m^2+m+1}{(m+1)(m^3-1)}$

3. $\frac{x+y}{x-y}$, $\frac{x-y}{x+y}$, and $\frac{16}{x^3-xy^2}$

Ans. $\frac{x^3+2x^2y+xy^2}{x(x^2-y^2)}$, $\frac{x^3-2x^2y+xy^2}{x(x^2-y^2)}$, *and* $\frac{16}{x(x^2-y^2)}$

ADDITION AND SUBTRACTION OF FRACTIONS.

50. In arithmetic we saw that $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ and $\frac{1}{3}+\frac{2}{3}=\frac{3}{3}=1$. Similarly in algebra $\frac{1}{a}+\frac{1}{a}=\frac{2}{a}$ and $\frac{1}{a}+\frac{x}{a}=\frac{1+x}{a}$. For if a quantity is divided into c parts and a parts are taken, $\frac{a}{c}$ represents the amount taken. Similarly if the quantity is divided into c parts and b parts are taken we have $\frac{b}{c}$ to represent the amount. Again if a quantity is divided into c parts and a parts are taken and b parts are also taken, the whole amount taken is $\frac{a+b}{c}$. By this we see that $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$.

It is further seen by similar reasoning that $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$

In addition or subtraction of fractions, reduce them if necessary to equivalent fractions having the lowest common denominator. Add or subtract the numerators as the case may be and write the sum or difference thus obtained over the common denominator.

The final result should be reduced to its simplest form.

Find the sum of $\frac{x^2+4}{2xy^2}$ and $\frac{y^2-7}{3x^2y}$

The lowest common denominator is $6x^2y^2$.

Multiplying both terms of the first fraction by $3x$ and both terms of the second by $2y$, we have

$$\frac{x^2+4}{2xy^2}+\frac{y^2-7}{3x^2y}=\frac{3x^3+12x}{6x^2y^2}+\frac{2y^3-14y}{6x^2y^2}.$$

$$=\frac{3x^3+12x+2y^3-14y}{6x^2y^2}.$$

Subtract $\frac{4m-1}{2m}$ from $\frac{6a-2}{3a}$

The lowest common denominator is $6am$.

$$\therefore \frac{6a-2}{3a}-\frac{4m-1}{2m}=\frac{12am-4m}{6am}-\frac{12am-3a}{6am}.$$

$$=\frac{12am-4m-(12am-3a)}{6am}.$$

$$=\frac{12am-4m-12am+3a}{6am}.$$

$$=\frac{3a-4m}{6am}.$$

NOTE. Remember that the line of the fraction acts as a vinculum, and it is therefore well to enclose the numerator of a fraction preceded by a — sign in a parenthesis.

Find the sum of $\frac{4a}{3b}$, $\frac{5}{6b}$, and $\frac{2a+b}{6b^2}$.

The lowest common denominator is $6b^2$.

$$\therefore \frac{4a}{3b}+\frac{5}{6b}+\frac{2a+b}{6b^2}=\frac{8ab}{6b^2}+\frac{5b}{6b^2}+\frac{2a+b}{6b^2}.$$

$$=\frac{8ab+5b+2a+b}{6b^2}.$$

$$=\frac{8ab+6b+2a}{6b^2}.$$

reducing

$$=\frac{4ab+3b+a}{3b^2}.$$

EXAMPLES FOR PRACTICE.

Simplify the following:

1. $\frac{3}{5xy^2} - \frac{1}{2x^2y}$. Ans. $\frac{6x - 5y}{10x^2y^2}$

2. $\frac{n - 4m}{24m} + \frac{m + 5n}{30n}$. Ans. $\frac{4m^2 + 5n^2}{120mn}$

3. $\frac{x^2 + 1}{3x^2} - \frac{6x^2 + 1}{12x^2} + \frac{y - 2}{6y}$. Ans. $\frac{4xy - y - 4x^2}{12x^2y}$

Let us now take a few examples in which the denominators are polynomials.

Simplify $\frac{1}{a - a^2} + \frac{1}{a + a^2}$

The L. C. D. is $a(1 - a^2)$. The denominator of the first fraction is contained in this $1 + a$ times, that of the second $1 - a$ times.

$$\therefore \frac{1}{a - a^2} + \frac{1}{a + a^2} = \frac{1 + a}{a(1 - a^2)} + \frac{1 - a}{a(1 - a^2)}$$
$$= \frac{1 + a + 1 - a}{a(1 - a^2)}$$
$$= \frac{2}{a(1 - a^2)}$$

Simplify $\frac{x + y}{x - y} - \frac{4xy}{x^2 - y^2} - \frac{x - y}{x + y}$

The L. C. D. is $x^2 - y^2$.

$$\therefore \frac{x + y}{x - y} - \frac{4xy}{x^2 - y^2} - \frac{x - y}{x + y}$$
$$= \frac{(x + y)^2}{x^2 - y^2} - \frac{4xy}{x^2 - y^2} - \frac{(x - y)^2}{x^2 - y^2}$$
$$= \frac{x^2 + 2xy + y^2 - 4xy - (x^2 - 2xy + y^2)}{x^2 - y^2}$$
$$= \frac{x^2 + 2xy + y^2 - 4xy - x^2 + 2xy - y^2}{x^2 - y^2}$$
$$= 0 \text{ as the sum of the terms of the numerator is } 0.$$

EXAMPLES FOR PRACTICE.

Simplify the following:

1. $\frac{1}{1 - a} + \frac{1}{1 + a}$ Ans. $\frac{2}{1 - a^2}$

2. $\frac{1}{m+7} - \frac{1}{m+8}$ Ans. $\frac{1}{m^2+15m+56}$

3. $\frac{x}{x-y} - \frac{y}{x+y}$ Ans. $\frac{x^2+y^2}{x^2-y^2}$

4. $\frac{x+y}{x-y} - \frac{x-y}{x+y}$ Ans. $\frac{4xy}{x^2-y^2}$

5. $\frac{x+y}{(x-y)^2} + \frac{2x}{x^2-y^2}$ Ans. $\frac{3x^2+y^2}{(x+y)(x-y)^2}$

6. $\frac{x}{xy-y^2} - \frac{1}{x-y} - \frac{1}{y}$ Ans. 0

7. $\frac{m+n}{m-n} + \frac{m-n}{m+n} + 2$ Ans. $\frac{4m^2}{m^2-n^2}$

8. $\frac{a-3}{a-2} - \frac{a+1}{a+5} + \frac{a+13}{a^2+3a-10}$ Ans. $\frac{4a}{a^2+3a-10}$

It is often much more convenient to change the form of a fraction so that it will be easier to add or subtract. For example:

Simplify $\frac{3}{x-y} + \frac{2y+x}{y^2-x^2}$

As we may change the sign before the fraction *and* the sign of the denominator without changing the value of the fraction, we may write the above example thus:

$$\frac{3}{x-y} - \frac{2y+x}{x^2-y^2}$$

The L. C. D. is now $x^2 - y^2$.

$$\therefore \frac{3}{x-y} - \frac{2y+x}{x^2-y^2} = \frac{3(x+y)-(2y+x)}{x^2-y^2}$$

$$= \frac{3x+3y-2y-x}{x^2-y^2}$$

$$= \frac{2x+y}{x^2-y^2}$$

Simplify $\frac{1}{(a-b)(a-c)} - \frac{1}{(b-a)(b-c)} - \frac{1}{(c-a)(c-b)}$

We may, without altering the value, change the sign of one of the factors in the second denominator *and* the sign before that fraction, and also we may change the sign of both factors in the third denominator. Our expression thus becomes:

$$\frac{1}{(a-b)(a-c)} + \frac{1}{(a-b)(b-c)} - \frac{1}{(a-c)(b-c)}$$

The L. C. D. is now $(a-b)\ (a-c)\ (b-c)$.

Hence the expression

$$= \frac{(b-c)+(a-c)-(a-b)}{(a-b)\ (a-c)\ (b-c)}$$

$$= \frac{b-c+a-c-a+b}{(a-b)\ (a-c)\ (b-c)}$$

$$= \frac{2b-2c}{(a-b)\ (a-c)\ (b-c)}$$

$$= \frac{2(b-c)}{(a-b)\ (a-c)\ (b-c)}$$

$$= \frac{2}{(a-b)\ (a-c)}$$

EXAMPLES FOR PRACTICE.

Simplify the following:

1. $\dfrac{1}{a^2-ab} - \dfrac{1}{b^2-a^2}$ Ans. $\dfrac{2a+b}{a\,(a^2-b^2)}$

2. $\dfrac{5x+1}{3x-3} - \dfrac{3x-1}{2-2x}$ Ans. $\dfrac{19x-1}{6\,(x-1)}$

3. $\dfrac{3}{2-m} + \dfrac{5}{m-3} + \dfrac{1}{m^2-5m+6}$ Ans. $\dfrac{2m}{m^2-5m+6}$

MULTIPLICATION OF FRACTIONS.

51. In arithmetic we saw that $\frac{2}{3} \times \frac{5}{11}$ signified two thirds of $\frac{5}{11}$ or $\frac{10}{33}$. In algebra $\frac{a}{b} \times \frac{c}{d}$ has a similar signification resulting in $\frac{ac}{bd}$, this being the result of multiplying $\frac{c}{d}$ by a, and dividing the product, $\frac{ac}{d}$, by b obtaining $\frac{ac}{bd}$.

Hence we have the following rule:

Multiply the numerators together for the numerator of the product and the denominators for the denominators of the product.

Reduce mixed quantities to fractional form before applying the rule.

Cancel common factors before multiplying.

Let us illustrate by a few examples:

Multiply $\frac{6a^2b}{7c^3d^2}$ by $\frac{14a^3d^4}{27a^3b^5}$.

$$\frac{6a^2b}{7c^3d^2} \times \frac{14a^3d^4}{27a^3b^5} = \frac{2 \cdot 3 \cdot 2 \cdot 7a^2bc^3d^4}{7 \cdot 3 \cdot 3 \cdot 3a^3b^5c^3d^2}$$

$$= \frac{2 \cdot 2d^2}{3 \cdot 3ab^4c}$$

$$= \frac{4d^2}{9ab^4c}$$

Multiply $\frac{a^2 - 2a}{a^2 - 2a - 3}$, $\frac{a^2 - 9}{a^2 - a}$, and $\frac{a^2 + a}{a^2 + a - 6}$

$$\frac{a^2 - 2a}{a^2 - 2a - 3} \times \frac{a^2 - 9}{a^2 - a} \times \frac{a^2 + a}{a^2 + a - 6}$$

$$= \frac{a(a - 2)}{(a - 3)(a + 1)} \times \frac{(a + 3)(a - 3)}{a(a - 1)} \times \frac{a(a + 1)}{(a + 3)(a - 2)}$$

$$= \frac{a}{a - 1}$$

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $\frac{5x}{13y}$, $\frac{6z}{5x}$, and $\frac{5y}{8z}$ Ans. $\frac{15}{52}$

2. $\frac{a^2 - 16}{a^2 + 5a}$ and $\frac{a^2 - 25}{a^2 - 4a}$ Ans. $\frac{a^2 - a - 20}{a^2}$

3. $\frac{1 - m^2}{1 - n}$, $\frac{1 - n^2}{m - m^2}$, and $\frac{1}{1 + m}$ Ans. $\frac{1 + n}{m}$

DIVISION OF FRACTIONS.

52. Division is the reverse of multiplication. Suppose we investigate by finding what $\frac{a}{b} \div \frac{c}{d}$ equals. As the quotient $\times$ the divisor $=$ the dividend, the quantity required is evidently the quantity which when multiplied by $\frac{c}{d}$ will produce $\frac{a}{b}$

The required quantity is obviously $\frac{ad}{bc}$

Hence,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

The result is the same as if we multiplied by $\frac{d}{c}$. Hence the following rule:

Invert the divisor and multiply.

Reduce mixed quantities to fractional form.

To illustrate:

Divide $\frac{4x^2y}{5m^2}$ by $\frac{8xy^2}{15m}$

$$\frac{4x^2y}{5m^2} \div \frac{8xy^2}{15m} = \frac{4x^2y}{5m^2} \times \frac{15m}{8xy^2}$$
$$= \frac{3x}{2my}$$

Divide $\frac{a^2-9}{16}$ by $\frac{a^2+2a-3}{4}$

$$\frac{a^2-9}{16} \div \frac{a^2+2a-3}{4} = \frac{(a+3)(a-3)}{16} \times \frac{4}{(a+3)(a-1)}$$
$$= \frac{a-3}{4(a-1)}$$

EXAMPLES FOR PRACTICE.

Divide:

1. $\frac{18ax^3}{25by^3}$ by $\frac{6a^2x^4}{5b^2y^4}$ Ans. $\frac{3by^2}{5ax}$

2. $\frac{m^2-25m}{m^2+m-6}$ by $\frac{m^2-5m}{m^2-m-12}$ Ans. $\frac{m^2+m-20}{m-2}$

3. $9+\frac{5b^2}{a^2-b^2}$ by $3+\frac{5b}{a-b}$ Ans. $\frac{3a-2b}{a+b}$

COMPLEX FRACTIONS.

53. A **complex fraction** is one which contains a fraction in one or both of its terms. It may be regarded as a case in division, the numerator as the dividend and the denominator as the divisor.

To illustrate:

Reduce $\frac{a}{b+\frac{c}{d}}$ to its simplest form.

$$\frac{a}{b+\frac{c}{d}} = \frac{a}{\frac{bd+c}{d}} = a \times \frac{d}{bd+c} = \frac{ad}{bd+c}$$

Reduce $\dfrac{\frac{x}{x-y} - \frac{x}{x+y}}{\frac{y}{x-y} + \frac{x}{x+y}}$ to its simplest form.

The L. C. M. of $x + y$ and $x - y$ is $(x + y)\ (x - y)$.

Simplifying the numerator and denominator we have,

$$\frac{\frac{2xy}{x^2-y^2}}{\frac{x^2+y^2}{x^2-y^2}} = \frac{2xy}{x^2-y^2} \times \frac{x^2-y^2}{x^2+y^2} = \frac{2xy}{x^2+y^2} \qquad \text{Ans. } \frac{2xy}{x^2+y^2}$$

Reduce $\dfrac{1}{1+\frac{1}{1+\frac{1}{a}}}$ to simplest form.

$$\frac{1}{1+\frac{1}{1+\frac{1}{a}}} = \frac{1}{1+\frac{a}{a+1}} = \frac{a+1}{a+1+a} = \frac{a+1}{2a+1}$$

Reduce $\dfrac{1-\frac{x}{x+\frac{1}{y}}}{1-\frac{1}{y-\frac{1}{y}}}$ to its simplest form.

$$\frac{1-\frac{x}{x+\frac{1}{y}}}{1-\frac{1}{y-\frac{1}{y}}} = \frac{1-\frac{x}{\frac{xy+1}{y}}}{1-\frac{1}{\frac{y^2-1}{y}}} = \frac{1-\frac{xy}{xy+1}}{1-\frac{y}{y^2-1}} =$$

$$\frac{\frac{xy+1-xy}{xy+1}}{\frac{y^2-1-y}{y^2-1}} = \frac{\frac{1}{xy+1}}{\frac{y^2-1-y}{y^2-1}} = \frac{1}{xy+1} \times \frac{y^2-1}{y^2-y-1} =$$

$$\frac{y^2-1}{(xy+1)\ (y^2-y-1)} \qquad \text{Ans. } \frac{y^2-1}{(xy+1)\ (y^2-y-1)}$$

EXAMPLES FOR PRACTICE.

1. $$\frac{x - y + \dfrac{y^2}{x+y}}{\dfrac{x+y}{x-y} - \dfrac{y(2x+y)}{x^2-y^2}}$$ Ans. $x - y$

2. $$\frac{a + \dfrac{b^2 - ab}{a-b}}{(a^2 - b^2) - \dfrac{a^3 + ab^2 - 2b^3}{a+b}}$$ Ans. $\dfrac{a+b}{b(a-b)}$

3. $$\frac{1 - \dfrac{3xy(x-y)}{x^3 - y^3}}{1 + \dfrac{xy}{x^2 + xy + y^2}}$$ Ans. $\dfrac{(x-y)^2}{(x+y)^2}$

SECTION 6.

ALGEBRA.

PART II.

SIMPLE EQUATIONS.

1. An equation is a statement of the equality of two expressions. The first of these expressions, i. e., to the left of the sign of equality, is called the *First Member*, the second, i. e., on the right of the sign of equality, the *Second Member* of the equation. The *sides* of an equation are its two members.

Thus in the equation $a + 3 = x - 1$, the first member is $a + 3$, the second is $x - 1$. The equation states that a increased by 3 is equal to x decreased by 1, i. e., x is equal to a increased by 4.

An equation in which all the known quantities are represented by numbers is called a **numerical equation.** One in which the known quantities are represented by letters is called a **literal** equation. The following are examples of numerical and literal equations respectively: $7x - 3 = 14$ and $3x + 5a = 14b - bx$. An **identical equation** is one in which both members are equal, no matter what values are given to the letters involved; as $x^2 - a^2 = (x + a)(x - a)$.

The **degree** of an equation which contains but one unknown quantity is denoted by the highest power of that unknown quantity in the equation. Thus $x + 15 = 4x - 5$ and $a^2x - 4 = b$ are equations of the *first* degree. $x^2 + 15 = 4x - 5$, and $ax^2 - 11 = bc + d$, are equations of the *second* degree, etc.

2. The **root** of an equation containing but one unknown quantity, is the value of that unknown quantity; or, it is the value which, substituted for that unknown quantity causes the equation to become identical. Take the equation $4x - 3 = x + 12$. If 5 is substituted for x we have $20 - 3 = 5 + 12$, i. e., the equation becomes identical each side equalling 17. Thus the root of the equation, or the value of x, is 5.

NOTE.—Simple equations have but one root. Equations of higher degree possess a greater number.

The **solution** of an equation is the process of finding its roots.

An equation is *satisfied*, or a root *verified*, when on substituting the value of the root in place of its symbol, the equation becomes identical.

3. The following operations are based on the axiom or self-evident truth that,

If the same operations be performed upon equals the results will be equal, and therefore

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

TRANSPOSITION.

4. We may transpose any term from one side of an equation to the other *by changing its sign*.

To illustrate observe the following equation:

$$x + a = b.$$

Now subtracting a from each side we do not effect the equality for subtracting an equal amount from two equal amounts does not affect their equality.

Hence,

$$x + a - a = b - a.$$

or,

$$x = b - a.$$

That is we have moved the $+ a$ on the left hand side of the equation to the right hand side, changing its sign from $+$ to $-$ so that it appears as $- a$.

If the equation is $x - a = b$ we add a to each side, thus:

$$x - a + a = b + a.$$

or,

$$x = b + a.$$

That is the $- a$ on the left hand side of the equation becomes $+ a$ on the right hand side.

If we have an equation with identical terms with the same sign on each side of the equation, those terms cancel. This is readily seen by the following:

$$x + a - b = a + c.$$

Subtracting a from each side we have:

$$x + a - b - a = a + c - a.$$

or,

$$x - b = c.$$

If we change all the signs of an equation we do not destroy the equality.

Take the equation $x - y = a - b$.

Transposing each term on the right hand side we have:

$$x - y - a + b = 0.$$

Transposing x and $-y$ to the right hand side we have:

$$-a + b = -x + y.$$

or,

$$b - a = y - x$$

or,

$$y - x = b - a,$$

which is the same as the original, except that every sign is changed.

Similar terms should be united, for example:

$$x + y - 3x = a - 5x.$$

This may be written thus with all the x's on one side:

$$x + y - 3x + 5x = a.$$

or as,

$$x - 3x + 5x + y = a,$$

or, $3x + y = a.$

EXAMPLES FOR PRACTICE.

1. $3a + 24 + 2y = 64 - 5a + 11 - 4y + 6a$, (*transposing*), $3a + 5a - 6a + 2y + 4y = 64 - 24 + 11$, (*uniting terms*), $2a + 6y = 51$.

2. $18m - 6n + 4 - 17n = 44 - 22m + 62n$, *as before*), $40m - 85n = 40$.

3. $24p - 6p + 6 - 7r = 14p - 10r - 32$, (*becoming*), $4p + 3r = -38$.

SOLUTION OF SIMPLE EQUATIONS.

5. Solve the equation $8x - 11 = x + 3$.

Transposing the unknown quantities to the first member of the equation, i.e., to the left hand side, and the known quantities to the second member we have:

$$8x - x = 11 + 3,$$

and uniting similar terms we have:

$$7x = 14.$$

Dividing each member by 7 we have:

$$x = 2.$$

If we substitute 2 for x in the given equation we have:

$$16 - 11 = 2 + 3,$$

which gives $5 = 5$. Thus our equation is verified.

We may now formulate the following rules:

Transpose unknown terms to the first member, and known terms to the second.

Unite the similar terms and divide by the coefficient of the unknown quantity.

Solve the equation, $4 + 27a = 42a - 41$.

(*Transposing*), $27a - 42a = -41 - 4$.

(*Uniting terms*), $-15a = -45$.

(*Dividing by* 15), $-a = -3$.

(*Changing all signs*), $a = 3$. *Ans.*

EXAMPLES FOR PRACTICE.

1. $7x + 5 = -30$.
2. $3x - 8 = 5x + 8$.
3. $18 - 7x = 18x - 7$.
4. $32x - 15 = 7 + 65x$.

SOLUTION OF EQUATIONS CONTAINING FRACTIONS.

6. Solve the equation $\frac{3x}{2} - \frac{5}{4} = \frac{x}{3} - \frac{4}{6}$

The L. C. M. of 2, 3, 4 and 6 is 12. Multiplying each term of the equation by 12 we have:

$$\frac{36x}{2} - \frac{60}{4} = \frac{12x}{3} - \frac{48}{6}$$

which is equal to

$$18x - 15 = 4x - 8,$$

or,

$$18x - 4x = 15 - 8,$$

whence

$$14x = 7,$$

and

$$x = \frac{1}{2}$$

Therefore, to clear an equation of fractions:

Multiply each term by the lowest common multiple of the denominators.

To take a further illustration:

Solve the equation $\frac{2a}{3} - \frac{5}{4} = \frac{5a}{6} - \frac{9}{8}$

The L. C. M. of 3, 4, 6 and 8 is 24. Multiplying each term of the equation by 24 we have:

$$16a - 30 = 20a - 27$$
$$16a - 20a = 30 - 27$$
$$-4a = 3$$
$$a = -\frac{3}{4} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. $\frac{3x}{4} - \frac{5x}{6} + \frac{1}{18} = 0$ Ans. $x = \frac{2}{3}$

2. $\frac{2x}{5} - \frac{9x}{20} - \frac{7x}{10} = \frac{5}{4}$ Ans. $x = -1\frac{2}{3}$

3. $\frac{3}{x} - \frac{7}{2x} = \frac{7}{12} - \frac{5}{3x}$ Ans. $x = 2$.

ILLUSTRATIVE EXAMPLES.

Solve the equation $\frac{3a-1}{4} - \frac{4a-5}{5} = 4 + \frac{7a+5}{10}$

Multiplying through by 20, the L. C. M. of 4, 5 and 10, we have:

$$15a - 5 - (16a - 20) = 80 + 14a + 10$$
$$15a - 5 - 16a + 20 = 80 + 14a + 10$$
$$15a - 16a - 14a = 80 + 10 + 5 - 20$$
$$-15a = 75$$
$$-a = 5.$$
$$a = -5. \quad \text{Ans.}$$

NOTE. When a fraction, whose numerator is a polynomial, is preceded by a minus sign, the sign of each term of the numerator should be carefully changed when the denominator is removed. In such a case, the numerator may be enclosed in a parenthesis, as shown in the above example.

EXAMPLE FOR PRACTICE.

$7x - \frac{11x-3}{4} = 3x + 7.$ Ans. $x = 5$.

ILLUSTRATIVE EXAMPLES.

1. Solve the equation $\frac{2}{x-1} - \frac{3}{x+1} - \frac{1}{x^2-1} = 0$.

Multiplying each term by $x^2 - 1$, the L. C. M. of the denominators, we have:

$$2(x+1) - 3(x-1) - 1 = 0$$
$$2x + 2 - 3x + 3 - 1 = 0$$
$$2x - 3x = -2 - 3 + 1$$
$$-x = -4$$
$$x = 4. \quad \text{Ans.}$$

2. Solve the equation $\frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}$

Multiplying each term by 15 we have:

$$6x + 1 - \frac{30x - 60}{7x - 16} = 6x - 3.$$

Transposing and uniting terms:

$$4 = \frac{30x - 60}{7x - 16}$$

Multiplying by $7x - 16$,

$$28x - 64 = 30x - 60$$
$$28x - 30x = 64 - 60$$
$$-2x = 4$$
$$x = -2. \quad \text{Ans.}$$

NOTE. When the denominators are partly monomial and partly polynomial, it often simplifies matters to clear *partially* of the fraction at first, multiplying by a quantity which will remove the *monomial* denominators.

EXAMPLES FOR PRACTICE.

1. $\frac{5a-2}{a(a-1)} = \frac{5a+7}{a^2-1}$ Ans. $a = -\frac{1}{2}$

2. $\frac{1}{x+1} + \frac{2}{x+2} = \frac{3}{x+3}$ Ans. $x = -\frac{2}{3}$

*3. $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-3}{x-4} - \frac{x-4}{x-5}$ Ans. $x = \frac{7}{2}$

*NOTE: To obtain the L. C. M. of the denominators multiply the denominators together.

SOLUTION OF LITERAL EQUATIONS.

7. ILLUSTRATIVE EXAMPLES.

1. Solve the equation: $2ax - 3b = x + c - 3ax$.

Transposing and uniting, $5ax - x = 3b + c$

Factoring the first member, $x(5a - 1) = 3b + c$

Dividing by $5a - 1$ $$x = \frac{3b + c}{5a - 1}.$$ Ans.

2. Solve the equation $(b - cx)^2 - (a - cx)^2 = b(b - a)$.

Performing the operations indicated,

$$b^2 - 2bcx + c^2x^2 - (a^2 - 2acx + c^2x^2) = b^2 - ab.$$
$$b^2 - 2bcx + c^2x^2 - a^2 + 2acx - c^2x^2 = b^2 - ab.$$
$$2acx - 2bcx = a^2 - ab.$$

Factoring both members, $2cx(a - b) = a(a - b.)$

Dividing by $2c(a - b)$, $$x = \frac{a(a - b)}{2c(a - b)}$$

$$x = \frac{a}{2c}$$ Ans.

EXAMPLES FOR PRACTICE.

1. $2ax + d = 3c - bx$. Ans. $x = \frac{3c - d}{2a + b}$

2. $\frac{x - m}{n} - \frac{x + n}{m} + 2 = 0$. Ans. $x = m - n$

SOLUTION OF EQUATIONS INVOLVING DECIMALS.

8. ILLUSTRATIVE EXAMPLE.

Solve the equation $.2x - .01 - .03x = .113x + .161$.

Changing decimals to common fractions:

$$\frac{2x}{10} - \frac{1}{100} - \frac{3x}{100} = \frac{113x}{1{,}000} + \frac{161}{1{,}000}$$

Clearing of fractions:

$$200x - 10 - 30x = 113x + 161$$
$$57x = 171$$
$$x = 3.$$ Ans.

Or, we may solve the equation as follows:

Transposing, $.2x - .03x - .113x = .01 + .161$

Uniting terms, $.057x = .171$

Dividing by .057, $x = 3.$ Ans.

EXAMPLE FOR PRACTICE.

Solve the equation:

$.23x - 2.05 = .02x - 1.882$. Ans. $x = .8$.

PROBLEMS.

9. The following leads to simple equations which contain but one unknown quantity.

The solution of a problem in algebra must depend in large measure upon the natural ingenuity of the student. No general rule, applying to all cases, can be given. The following suggestions however may be found useful:

(*a*.) Represent the unknown quantity, or one of the unknown quantities, by one of the final letters in the alphabet.

(*b*.) Try to discover, from the given conditions, expressions for the other unknown quantities, if such exist, in the problem.

(*c*.) Equations should be formed in accordance with the conditions given in the problem.

(*d*.) Solve the equation thus formed.

ILLUSTRATIVE EXAMPLES.

1. What is the number which if four-sevenths of itself be added, the sum will equal twice the number diminished by 27?

Let x = the number.

Then $\frac{4x}{7}$ = four-sevenths of it,

And $2x$ = twice it.

By the conditions, $x + \frac{4x}{7} = 2x - 27$

$$7x + 4x = 14x - 189$$
$$-3x = -189$$

Whence, $x = 63$. Ans.

2. A is three times as old as B, and eight years ago he was seven times as old as B. What are their present ages?

Let x = B's age.

Then $3x$ = A's age.

Also $x - 8$ = B's age 8 years ago.

And $3x - 8$ = A's age 8 years ago.

By the conditions, $3x - 8 = 7\ (x - 8)$

$$3x - 8 = 7x - 56$$
$$-4x = -48$$

Whence, $x = 12$,

And $3x = 36$,

NOTE.— Do not be confused by the expression, "Let x = B's age." x can of course represent only an abstract number, —— the number of years in B's age.

3. A had twice as much money as B; but after giving B \$35, he had only one-third as much as B. How much had each at first?

Let x = what B had at first.

Then $2x$ = what A had at first.

Then, after giving B \$35, A had left $2x - 35$ dollars, while B had $x + 35$ dollars.

By the conditions therefore $x + 35 = 3\ (2x - 35)$

$$x + 35 = 6x - 105$$
$$-5x = -140$$

Whence, $x = 28$,

$2x = 56$,

EXAMPLES FOR PRACTICE.

1. What number is that whose double exceeds its half by 45? Ans. 30.

2. A is 62 years of age, and B is 36. How many years is it since A was three times as old as B? Ans. 23.

3. A gentleman left an estate of \$1,872 to be divided between his wife, three sons, and two daughters. The wife was to receive three times as much as either of the daughters, and each son one-half as much as each of the daughters. How much did each receive?

Ans. wife, \$864, a daughter, \$288, a son, \$144.

ILLUSTRATIVE EXAMPLES.

1. Divide a into two parts such that m times the first part shall be equal to n times the second.

Let . . . x = the first part.

Then . . . $a - x$ = the second part.

By the conditions, $mx = n\,(a - x)$

Or, $mx + nx = an$

Whence, $x = \frac{an}{m+n}$, the first part.

And $a - x = a - \frac{an}{m+n} = \frac{am}{m+n}$,

the second part. Ans.

2. A can do a piece of work in 8 days which B can perform in 10 days. In how many days can it be done by both working together?

Let x = number of days required.

Then $\frac{1}{x}$ = what both can do in one day.

Also $\frac{1}{8}$ = what A can do in one day.

And $\frac{1}{10}$ = what B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}$

$5x + 4x = 40$

$x = 4\frac{4}{9}$, $4\frac{4}{9}$ days, Ans.

EXAMPLE FOR PRACTICE.

A can do a piece of work in a hours, which B can do in b hours. In how many hours can it be done by both working together? $\frac{ab}{a+b}$ hours. Ans.

ILLUSTRATIVE EXAMPLES.

1. A sum of money, amounting to \$4.32, consists entirely of dimes and cents, there being in all 108 coins. How many are there of each kind?

Let x = the number of dimes.

Then $108 - x$ = number of cents.

Also $10x$ = the value of dimes in *cents*

By the conditions,

$10x + 108 - x = 432$

$9x = 324$

Whence, $x = 36$, number of dimes.

And $108 - x = 72$, number of cents. Ans.

2. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of its digits, the quotient is 5. Required the number.

Let x = the first digit.
Then $x + 2$ = the second.
And $2x + 2$ = the sum of the digits.

The number itself is equal to 10 times the first digit, plus the second, which is $10x + x + 2$, or $11x + 2$. Hence, by the conditions,

$$\frac{11x + 2 + 6}{2x + 2} = 5$$

$$11x + 8 = 10x + 10$$

Whence, $x = 2$

Therefore, $11x + 2 = 24$ 24 Ans.

EXAMPLES FOR PRACTICE.

The first digit of a number is three times the second; and if the number, increased by 3, be divided by the difference of its digits, the quotient is 16. Required the number. Ans. 93.

ILLUSTRATIVE EXAMPLES.

1. Two persons, A and B, 63 miles apart, start at the same time and travels towards each other. A travels 4 miles an hour, and B 3 miles an hour. How far will each have traveled when they meet?

Let, x = the distance A travels.
Then, $63 - x$ = the distance B travels.
Also, $\frac{x}{4}$ = the time A takes to go x miles.
And $\frac{63 - x}{3}$ = the time B takes to go $63 - x$ miles.

By the conditions, $\frac{x}{4} = \frac{63 - x}{3}$

$$3x = 252 - 4x$$
$$7x = 252$$
$$x = 36,$$
$$63 - x = 27,$$

2. Separate 41 into two parts such that one divided by the other may give 1 as a quotient and 5 as a remainder.

Let x = the divisor.

Then $41 - x$ = the dividend.

By the conditions, $\frac{41 - x}{x} = 1 + \frac{5}{x}$

$$41 - x = x + 5$$
$$-2x = -36$$
$$x = 18, \text{ the divisor.}$$
$$41 - x = 23, \text{ the dividend.}$$

Ans.

EXAMPLES FOR PRACTICE.

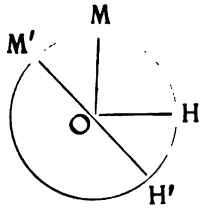
Separate 113 into two parts such that one divided by the other may give 2 as a quotient and 20 as a remainder.

Ans. 82, the dividend; 31, the divisor.

ILLUSTRATIVE EXAMPLE.

At what time between 3 and 4 o'clock are the hands of a watch opposite each other?

Let OM and OH represent the positions of the minute and hour hands at 3 o'clock, and OM′ and OH′ their positions when opposite each other.



Let x = the arc MHH′M′ over which the minute-hand has passed since 3 o'clock. Then $\frac{x}{12}$ = the arc HH′ over which the hour-hand has passed since 3 o'clock.

Also, the arc MH = 15 minute spaces.

And the arc H′M′ = 30 minute spaces.

Now, arc MHH′M′ = arc MH + arc H′M′ + arc HH′.

That is, $x = 15 + 30 + \frac{x}{12}$

$$12x = 540 + x$$
$$11x = 540$$
$$x = 49\frac{1}{11}$$

$49\frac{1}{11}$ minutes after 3 o'clock. Ans.

EXAMPLE FOR PRACTICE.

A, B, and C together can do a piece of work in 6 days; B's work is one-half of A's, and C's work is two-thirds of B's. How many days would it take each working alone?

Ans. A, 11 days; B, 22 days; C, 33 days.

SIMPLE EQUATIONS.

10. **Containing two Unknown Quantities.**

When a single equation contains *two* unknown quantities, as $x + y = 20$, the values of x and y cannot be ***definitely*** determined; because, if any value be assumed for x, we have a corresponding value for y.

Thus, if $x = 15$, then $15 + y = 20$, or $y = 5$,
if $x = 12$, then $12 + y = 20$, or $y = 8$, etc.

Hence, any of the pairs of values:

$$x = 15,\ y = 5;\ x = 12,\ y = 8;\ \text{etc.},$$

will satisfy a given equation.

Equations of this kind are called *indeterminate.*

But if we have the equations, $x + y = 20$ and $x - y = 10$, it is evident that the values $x = 15$, $y = 5$, are the only correct ones, since they satisfy both equations ***simultaneously.***

Simultaneous Equations are such as are satisfied by the ***same values*** of their unknown quantities.

Independent Equations are such as can be made to assume the same form.

Two unknown quantities require for their determination *two* independent, simultaneous equations, which may be solved by an operation called **Elimination,** which combines them so as to form an equation containing but *one* unknown quantity.

The three methods of elimination are:

1. By Addition or Subtraction.
2. By Substitution.
3. By Comparison.

ELIMINATION BY ADDITION OR SUBTRACTION.

11. **Rule.** ***Multiply the given equations by such numbers as will make the coefficients of one of the unknown quantities equal. Then add or subtract the resulting equations according as the equal coefficients have like or unlike signs.***

NOTE. Such multipliers should be used as will produce the least common multiple of the coefficients which are to be made equal.

ILLUSTRATIVE EXAMPLES.

1. Solve the equations $\begin{cases} 5x - 3y = 19 & (1.) \\ 7x + 4y = 2 & (2.) \end{cases}$

Multiplying (1) by 4, $20x - 12y = 76$

Multiplying (2) by 3, $21x + 12y = 6$

Adding these equations, $41x = 82$

$x = 2$

Substituting the value x in (1), $10 - 3y = 19$

$-3y = 9$

$y = -3$

Ans. $x = 2, y = -3$

This is an example of elimination by *addition*.

2. Solve the equations $\begin{cases} 15x + 8y = 1 & (1.) \\ 10x - 7y = -24 & (2.) \end{cases}$

Multiplying (1) by 2, $30x + 16y = 2$ (3.)

Multiplying (2) by 3, $30x - 21y = -72$ (4.)

Subtracting (4) from (3), $37y = 74$

$y = 2$

Substituting this value in (2), $10x - 14 = -24$

$10x = -10$

$x = -1$

Ans. $x = -1, y = 2.$

EXAMPLES FOR PRACTICE.

1. Solve the equations $\begin{cases} 7x + 2y = 31 \\ 3x - 4y = 23 \end{cases}$

Ans. $x = 5, y = -2$

2. Solve the equations $\begin{cases} 7b - 5a = -11 \\ 6a - b = 28 \end{cases}$

Ans. $a = 5, b = 2$

ELIMINATION BY SUBSTITUTION.

12. **Rule.** *Find the value of one of the unknown quantities in terms of the other from one of the given equations, and substitute this value for that quantity in the other equation.*

ILLUSTRATIVE EXAMPLE.

Solve the equations $\begin{cases} 7x - 3y = -62 & (1.) \\ 2y - 5x = 44 & (2.) \end{cases}$

Transposing $5x$ in (2), $2y = 5x + 44$

$$y = \frac{5x + 44}{2} \qquad (3.)$$

Substituting this in (1), $7x - 3\left(\frac{5x + 44}{2}\right) = -62$

or $$7x - \frac{15x + 132}{2} = -62$$

Clearing of fractions, $14x - 15x - 132 = -124$

$$-x = 8$$

$$x = -8$$

Substituting this value in (3) $y = \frac{-40 + 44}{2} = 2$

Ans. $x = -8, y = 2$

EXAMPLES FOR PRACTICE.

1. Solve, by method of substitution, the equations,

$\left.\begin{matrix} 3a - 4b = 2 \\ 2a - 5b = 6 \end{matrix}\right\}$ Ans. $a = -2, b = -2$

2. Solve, by method of substitution, the equations,

$\left.\begin{matrix} 7y - 3x = 139 \\ 2x + 5y = 91 \end{matrix}\right\}$ Ans. $x = -2, y = 19$

ELIMINATION BY COMPARISON.

13. **Rule.** *Find the value of the same unknown quantity in terms of the other, from each of the equations, and place these values equal to each other.*

ILLUSTRATIVE EXAMPLE.

Solve by method of comparison the equations,

$$\begin{cases} 2x - 5y = -16 & (1.) \\ 3x + 7y = 5 & (2.) \end{cases}$$

Transposing $-5y$ in (1), $2x = 5y - 16$

or $$x = \frac{5y - 16}{2} \qquad (3.)$$

Transposing $7y$ in (2) $3x = 5 - 7y$

or $$x = \frac{5 - 7y}{3}$$

Equating these values of x, $\frac{5y - 16}{2} = \frac{5 - 7y}{3}$

Clearing of fractions, $15y - 48 = 10 - 14y$

$$29y = 58$$

$$y = 2$$

Substituting this value in (3) $x = \frac{10 - 16}{2} = -3$

Ans. $x = -3$, $y = 2$

EXAMPLES FOR PRACTICE.

Solve the following equations by the method of comparison: —

(1.) $\begin{cases} 6x + 5y = -8 \\ 4x + 3y = -5 \end{cases}$ Ans. $x = -\frac{1}{2}$, $y = -1$

(2.) $\begin{cases} 7y - 12x = 17 \\ 8x + 11y = 20 \end{cases}$ Ans. $x = -\frac{1}{4}$, $y = 2$

(3.) $\begin{cases} 15x + 6y = -7 \\ 8y - 21x = 18 \end{cases}$ Ans. $x = -\frac{2}{3}$, $y = \frac{1}{2}$

14. **Simple Equations, containing more than two unknown quantities.**

When we have *three* simple equations, containing in all *three* unknown quantities, we may obtain two of their values by combining for purposes of elimination two of the equations, and then combining the third with either of the others. By solving the

equations thus derived (they should contain but two unknown quantities) we find the values of two of the unknown quantities. Substitute these values in either of the original equations and the value of the third quantity may be obtained.

The same principle should be followed when the number of equations and unknown quantities is greater than three.

It is usually most convenient to employ the method of elimination by addition or subtraction.

ILLUSTRATIVE EXAMPLES.

1. Solve the equations

$$6x - 4y - 7z = 17 \quad (1.)$$
$$9x - 7y - 16z = 29 \quad (2.)$$
$$10x - 5y - 3z = 23 \quad (3.)$$

Multiplying (1) by 3, $18x - 12y - 21z = 51$
Multiplying (2) by 2, $18x - 14y - 32z = 58$

Subtracting, $2y + 11z = -7 \quad (4.)$

Multiplying (1) by 5, $30x - 20y - 35z = 85 \quad (5.)$
Multiplying (3) by 3, $30x - 15y - 9z = 69 \quad (6.)$

Subtracting (5) from (6) $5y + 26z = -16 \quad (7.)$

Multiplying (4) by 5, $10y + 55z = -35$
Multiplying (7) by 2, $10y + 52z = -32$

Subtracting, $3z = -3$
$z = -1$
Substituting in (4), $2y - 11 = -7$
$\therefore y = 2$

Substituting the values of y and z in (1,)

$$6x - 8 + 7 = 17$$
$$\therefore x = 3$$

Ans. $x = 3, y = 2, z = -1$

2. Solve the equations

$$\begin{cases} a + x + y = 6 & (1.) \\ x + y + z = 7 & (2.) \\ y + z + a = 8 & (3.) \\ z + a + x = 9 & (4.) \end{cases}$$

NOTE. *Sometimes the operation may be shortened by the aid of the following artifice.*

Add the given equations and simplify,

$$a + x + y + z = 10 \qquad (5.)$$

Subtracting (2) from (5), $a = 3$
Subtracting (3) from (5), $x = 2$
Subtracting (4) from (5), $y = 1$
Subtracting (1) from (5), $z = 4$

Ans.

EXAMPLES FOR PRACTICE.

1. Solve the equations

$$\left.\begin{aligned} x + y &= 2 \\ y + z &= -1 \\ z + x &= 3 \end{aligned}\right\} \qquad \text{Ans.} \quad \left\{\begin{aligned} x &= 3 \\ y &= -1 \\ z &= 0 \end{aligned}\right.$$

2. Solve the equations

$$\left.\begin{aligned} 4x - 3y + 2z &= 40 \\ 5x + 9y - 7z &= 47 \\ 9x + 8y - 3z &= 97 \end{aligned}\right\} \qquad \text{Ans.} \quad \left\{\begin{aligned} x &= 10 \\ y &= 2 \\ z &= 3 \end{aligned}\right.$$

3. Solve the equations

$$\left.\begin{aligned} ax + a^2y &= 2 \\ a^2y + a^3z &= 2 \\ a^3z + a^4x &= a^3 + 1 \end{aligned}\right\} \qquad \text{Ans.} \quad \left\{\begin{aligned} x &= \frac{1}{a} \\ y &= \frac{1}{a^2} \\ z &= \frac{1}{a^3} \end{aligned}\right.$$

4. Solve the equations,

$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 1 \\ \frac{1}{y} + \frac{1}{z} &= \frac{3}{2} \\ \frac{1}{z} + \frac{1}{x} &= 2 \end{aligned}\right\} \qquad \text{Ans.} \quad \left\{\begin{aligned} x &= \frac{4}{3} \\ y &= 4 \\ z &= \frac{4}{5} \end{aligned}\right.$$

PROBLEMS.

15. Leading to simple equations containing more than one unknown quantity.

ILLUSTRATIVE EXAMPLES.

1. Divide 81 into two parts such that three-fifths the greater shall exceed five-ninths the less by seven.

Let x = the greater part,
and y = the lesser.

By the conditions, $\begin{cases} x + y = 81 \\ \frac{3x}{5} = \frac{5y}{9} + 7 \end{cases}$

Solving these equations, $x = 45$, $y = 36$ Ans.

2. If three be added to both numerator and denominator of a fraction, its value is two-thirds; and if two be subtracted from both numerator and denominator, its value is one-half. Required the fraction.

Let x = the numerator,
and y = the denominator.

By the conditions, $\begin{cases} \frac{x+3}{y+3} = \frac{2}{3} \\ \frac{x-2}{y-2} = \frac{1}{2} \end{cases}$

Solving these equations, $x = 7$, $y = 12$

Therefore the fraction is $\frac{7}{12}$ Ans.

EXAMPLES FOR PRACTICE.

If the numerator of a fraction be doubled and the denominator increased by 7, its value is $\frac{2}{3}$; and if the denominator be doubled and the numerator increased by 2, the value is $\frac{3}{5}$. Required the fraction. Ans. $\frac{4}{5}$

ILLUSTRATIVE EXAMPLES.

A sum of money was divided equally between a certain number of persons. Had there been three more, each would have received \$1.00 less; had there been six less, each would have received \$5.00 more. How many persons were there and how much did each receive?

Let x = the number of persons,
and y = what each received.
Then xy = the sum divided.

By the conditions, $\begin{cases} (x+3)\ (y-1) = xy \\ (x-6)\ (y+5) = xy \end{cases}$

Solving these equations, $x = 12,\ y = 5$ Ans.

EXAMPLE FOR PRACTICE.

If B gives A \$5.00, they will have equal amounts; but if A gives B \$15.00, B will have seven-thirds as much as A. How much money has each? Ans. A, \$45.00; B, \$55.00.

ILLUSTRATIVE EXAMPLE.

The sum of the digits of a number of three figures is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the order of the digits will be reversed. Required the number.

Let x = the first digit.
y = the second.
z = the third.

Then $100x + 10y + z$ = the number.
And $100z + 10y + x$ = the number with its digits reversed.
By the conditions,

$$x + y + z = 13$$

$$\frac{100x + 10y + z - 8}{y + z} = 25$$

and $100x + 10y + z + 99 = 100z + 10y + x.$

Solving these equations, $x = 2,\ y = 8,\ z = 3$
Therefore, the number is 283. Ans.

EXAMPLE FOR PRACTICE.

If a number of two figures be divided by the sum of its digits, the quotient is 7; and if the digits be reversed, the quotient of the resulting number, increased by 6, divided by the sum of its digits is 5. Required the number. Ans. 42.

ILLUSTRATIVE EXAMPLES.

1. A crew can row 10 miles in 50 minutes down stream and 12 miles in an hour and a half against the stream. Find the rate

in miles per hour rowing against the current, and what the rate would be in still water.

Let x = the crew's rate in still water,

and y = the rate of the current.

Then $x + y$ = the rate rowing down stream,

and $x - y$ = the rate rowing up stream.

Since the distance divided by the rate gives the time, we have, by the conditions,

$$\begin{cases} \frac{10}{x+y} = \frac{5}{6} \\ \frac{12}{x-y} = \frac{3}{2} \end{cases}$$

Solving these equations, $x = 10$, $y = 2$. Ans.

2. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B the train would have been only one hour late. What was the rate of the train before the accident?

Let $3x$ = the rate of train before accident.

Then x = rate after accident.

Let y = the distance from B to the point of detention.

By the conditions, $\frac{y}{x} = \frac{y}{3x} + 5$.

$$\frac{y-60}{x} = \frac{y-60}{3x} + 1.$$

Solving these equations, $x = 10$, and the rate before the accident was 30 miles per hour. Ans.

EXAMPLE FOR PRACTICE.

A and B are building a fence 126 feet long. After 3 hours A leaves off, and B finishes the work in 14 hours. If 7 hours had occurred before A left off, B would have finished the work in $4\frac{2}{3}$ hours. How many feet does each build in an hour? Ans. A, 8; B, 6.

INVOLUTION.

16. The process of raising a quantity to any desired power is called **involution.** As is already evident it is effected by taking the quantity as a factor a number of times equal to the exponent of the required power.

If the quantity thus taken as a factor is positive, it is readily perceived that all its powers are positive, but if the quantity is negative, all its *even* powers will be positive, while all its odd powers will be negative.

For example:

$(-x)^2 = (-x) \times (-x) = +x^2$

$(-x)^3 = (-x) \times (-x) \times (-x) = -x^3$

$(-x)^4 = (-x) \times (-x) \times (-x) \times (-x) = +x^4$

$(-x)^5 = (-x) \times (-x) \times (-x) \times (-x) \times (-x)$
$= -x^5$, and so on.

We may sum up the preceding thus:

The even powers of any quantity are positive; the odd powers have the same sign as the quantity itself.

INVOLUTION OF MONOMIALS.

17. Find the value of $(4a^2b)^3$

$(4a^2b)^3 = 4a^2b \times 4a^2b \times 4a^2b = 64a^6b^3$

Find the value of $(-2x^3)^5$

$(-2x^3)^5 = (-2x^3) \times (-2x^3) \times (-2x^3) \times -2x^3) \times (-2x^3) = -32x^{15}$

The preceding process may be stated as follows:

Raise the numerical coefficient to the required power and multiply the exponent of each letter by the exponent of the required power.

If the power is even give it the positive sign, if odd give it the sign of the quantity itself.

A fraction is raised to any power by raising *both* numerator and denominator to the required power.

EXAMPLES FOR PRACTICE.

1. $(a^2bc^3)^2$ Ans. $a^4b^2c^6$
2. $(-3a^6x^m)^3$ Ans. $-27a^{18}x^{3m}$
3. $(-5x^2y^3)^4$ Ans. $625x^8y^{12}$
4. $(a^n b^2)^m$ Ans. $a^{mn}b^{2m}$
5. $\left(-\frac{4x^2}{ab^3}\right)^4$ Ans. $\frac{256x^8}{a^4b^{12}}$

SQUARE OF A POLYNOMIAL.

18. By multiplying we find:

$$\begin{array}{r} a + b + c \\ a + b + c \\ \hline a^2 + ab + ac \\ + ab \quad + b^2 + bc \\ + ac \quad + bc + c^2 \\ \hline a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \end{array}$$

For convenience this is usually written in the following order:

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

In the same manner we find that:

$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$

Hence the following rule for squaring a polynomial:

Write the square of each term and twice the product of each term by its succeeding terms.

Square $2a^2 - 3a - 4$

The square of the terms are respectively $4a^4$, $9a^2$ and 16. The products of the first term by the second and by the third are $-6a^3$ and $-8a^2$. Multiplying these products by 2 we have $-12a^3$ and $-16a^2$. Twice the product of the second term and the term succeeding it is $24a$. Hence we have,

$$(2a^2 - 3a - 4)^2 = 4a^4 + 9a^2 + 16 - 12a^3 - 16a^2 + 24a$$
$$= 4a^4 - 12a^3 - 7a^2 + 24a + 16$$

EXAMPLES FOR PRACTICE.

Square the following:

1. $x + y - z$ Ans. $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$
2. $2m^2 - m - 3$ Ans. $4m^4 - 4m^3 - 11m^2 + 6m + 9$
3. $4x^2 + xy^2 - 3y^4$
   Ans. $16x^4 + 8x^3y^2 - 23x^2y^4 - 6xy^6 + 9y^8$
4. $3a^3 - 2a^2 - a + 4$
   Ans. $9a^6 - 12a^5 - 2a^4 + 28a^3 - 15a^2 - 8a + 16$

CUBE OF A BINOMIAL.

19. We find by multiplication:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$a + b$$
$$a^3 + 2a^2b + ab^2$$
$$a^2b + 2ab^2 + b^3$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a - b$$
$$a^3 - 2a^2b + ab^2$$
$$-a^2b + 2ab^2 - b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

That is,

The cube of the sum of two quantities is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

The cube of the difference of two quantities is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

ILLUSTRATIVE EXAMPLES.

1. Find the cube of $a + 2b$

$$(a+2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$$
$$= a^3 + 6a^2b + 12ab^2 + 8b^3 \quad \text{Ans.}$$

2. Find the cube of $2m - 3n^2$

$$(2m - 3n^2)^3 = (2m)^3 - 3(2m)^2(3n^2) + 3(2m)(3n^2)^2 - (3n^2)^3$$
$$= 8m^3 - 36m^2n^2 + 54mn^4 - 27n^6. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the cubes of the following:

1. $2x - 1$ Ans. $8x^3 - 12x^2 + 6x - 1$
2. $a + 5b$ Ans. $a^3 + 15a^2b + 75ab^2 + 125b^3$
3. $2m^3 - 3m$ Ans. $8m^9 - 36m^7 + 54m^5 - 27m^3$

20. The cube of a trinomial can be obtained by the above method, provided that two of its terms be enclosed in a parenthesis and regarded as a single term.

ILLUSTRATIVE EXAMPLE.

Find the cube of $a^2 - 2a - 1$

$$(a^2 - 2a - 1)^3 = [(a^2 - 2a) - 1]^3$$
$$= (a^2 - 2a)^3 - 3(a^2 - 2a)^2 + 3(a^2 - 2a) - 1$$
$$= a^6 - 6a^5 + 12a^4 - 8a^3 - 3(a^4 - 4a^3 + 4a^2) + 3(a^2 - 2a) - 1$$
$$= a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the cubes of the following:

1. $x + y - z$ Ans. $x^3 + 3x^2y - 3x^2z + 3xy^2 - 6xyz + 3xz^2 + y^3 - 3y^2z + 3yz^2 - z^3$

2. $a - b + 1$ Ans. $a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1$

ANY POWER OF A BINOMIAL.

21. By actual multiplication, we obtain:

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; etc.

$(a - b)^2 = a^2 - 2ab + b^2$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$; etc.

In these results we discern the following laws:

I. The number of terms exceeds the exponent of the binomial by *one*.

II. The exponent of a in the first term is identical with the exponent of the binomial; and thereafter *decreases* by one in each succeeding term.

III. b first appears in the second term, having 1 as its exponent which thereafter *increases by* 1 in each succeeding term.

IV. The coefficient of the first term is 1; and that of the second is the exponent of the binomial.

V. The coefficient of any term may be found by dividing by the number of preceding terms the product of the coefficient and the exponent of a in the term next preceding.

(NOTE. *The number of preceding terms is always equal to the exponent of b increased by one.*)

VI. If the principal sign of the binomial is *minus*, then the signs are alternately *plus* and *minus*, beginning with *plus*; [e. g. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$]

ILLUSTRATIVE EXAMPLES.

1. Expand $(a + b)^5$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \text{ Ans.}$$

In the above, as we remarked in Law I, the first term a has 5 for exponent, and the latter *decreases* by *one* in each succeeding term. Likewise, b begins with one for exponent, which *increases* to five at the last. With regard to coefficients, we have 5, the exponent of the binomial for coefficient of the second term; and to obtain the coefficient of the third, we multiply that of the second by the exponent of a in that term, and divide that result by the number of terms preceding the one sought.

2. Expand $(3m - n^2)^4 = [(3m) - (n^2)]^4$

$$= (3m)^4 - 4(3m)^3 (n^2) + 6(3m)^2 (n^2)^2 - 4(3m) (n^2)^3 + (n^2)^4$$
$$= 81m^4 - 108m^3n^2 + 54m^2n^4 - 12mn^6 + n^8. \text{ Ans.}$$

EVOLUTION.

22. A **root** of a quantity is one of the equal factors into which it is resolved.

Evolution is the process of finding any required root of a quantity. It is accomplished by finding a quantity which, on being raised to the proposed power, will produce the given quantity.

The **radical sign** is $\surd$. When it is prefixed to a quantity it indicates that some root of that quantity is to be found.

$\surd a$ indicates the *second* or *square* root of a.

$\sqrt[3]{a}$ indicates the *third* or *cube* root of a.

$\sqrt[4]{a}$ indicates the *fourth* root of a.

$\sqrt[5]{a}$ indicates the *fifth* root of a.

And so on.

The figure written over the radical sign is called the *index*. If no index is written the square root is understood.

EVOLUTION OF MONOMIALS.

23. Find the cube root of $a^6b^3c^9$.

We must find a number which, raised to the third power, will produce $a^6b^3c^9$. The required number is evidently a^2bc^3.

Therefore $\sqrt[3]{a^6b^3c^9} = a^2bc^3$

To give the above principle in words:

Any root of a monomial is obtained by dividing the exponent of each factor by the index of the required root.

We have observed that odd powers have the same sign as the quantity itself, while even powers have the positive sign.

Hence,

1. The odd roots of a quantity have the same sign as the quantity itself.

To illustrate:

$$\sqrt[3]{a^3} = a$$

and $$\sqrt[3]{-a^3} = -a$$

2. The even roots of a positive quantity are written positive or negative.

To illustrate:

$$\sqrt[4]{a^4} = a \text{ or } -a$$

a or $-a$ is usually written $\pm a$

3. The even roots of a negative quantity are impossible.

This is evident because *no quantity raised to an even power can produce a negative result.* Such roots are termed *imaginary* quantities.

We may now state the following rule for determining the roots of a monomial.

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root.

Give to any even root of a positive quantity the sign +, and to any odd root of a quantity the sign of the quantity itself.

Any root of a fraction may be determined by taking the required root of *each* of its terms.

Find the square root of $16a^6b^{2m}$

$$\sqrt{16a^6b^{2m}} = +4a^3b^m$$

Find the fifth root of $-\frac{32x^{10}}{243y^{15}}$

$$\sqrt[5]{-\frac{32x^{10}}{243y^{15}}} = -\frac{2x^2}{3y^3}$$

Find the square root of $9x^4y^2z^6$

By the rule, $\sqrt{9x^4y^2z^6} = +3x^2yz^3$ **Ans.**

Find the fifth root of $-32a^{10}b^{5m}$

$\sqrt[5]{-32a^{10}b^{5m}} = -2a^2b^m$ **Ans.**

EXAMPLES FOR PRACTICE.

1. $\sqrt{121a^{14}c^2}$
2. $\sqrt[3]{a^{3n+3}b^{9m-6}}$
3. $\sqrt{\dfrac{a^4}{81b^8c^4}}$

NOTE.— *To obtain the root of any fraction, extract the required root of each of its terms.*

SQUARE ROOT OF POLYNOMIALS.

24. Since $(x+y)^2 = x^2 + 2xy + y^2$, we know that $x+y$ must be the square root of $x^2 + 2xy + y^2$. What process will determine this to be true, when the latter alone is given?

$$\begin{array}{r|l} x^2 + 2xy + y^2 & x + y \\ \cline{2-2} x^2 & \\ \hline 2x + y \;\Big|\; 2xy + y^2 & \\ \quad\;\Big|\; 2xy + y^2 & \\ \hline \end{array}$$

The square root of the first term is x, which is the first term of the root. Subtracting its square from the given expression, the remainder is $2xy + y^2$, or $(2x + y)\,y$. Dividing the first term of the remainder by $2x$, or twice the first term of the root, we obtain y, the second term. Add this to $2x$ to obtain the complete divisor $2x + y$; multiply this by y, and the product $2xy + y^2$, subtracted from the remainder, completes the operation.

From the above process we derive the following rule:

The terms should be arranged according to the powers of some letter.

The square root of the *first* term will be the *first* term of the root; after finding it, subtract its square from the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the root and also to the divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found, for the next trial divisor.

ILLUSTRATIVE EXAMPLES.

1. Find the square root of $9x^4 - 30a^3x^2 + 25a^6$

$$\begin{array}{l|l} 9x^4 - 30a^3x^2 + 25a^6 & 3x^2 - 5a^3 \quad \text{Ans.} \\ 9x^4 & \\ \hline 6x^2 - 5a^3 & -30a^3x^2 + 25a^6 \\ & -30a^3x^2 + 25a^6 \end{array}$$

2. Find the square root of

$$12x^5 - 14x^3 + 1 - 8x^4 + 9x^6 + 4x$$

Arranging according to the descending powers of x,

$$\begin{array}{l|l} 9x^6 + 12x^5 - 8x^4 - 14x^3 + 4x + 1 & 3x^3 + 2x^2 - 2x - 1 \\ 9x^6 & \text{Ans.} \\ \hline 6x^3 + 2x^2 & 12x^5 - 8x^4 \\ & 12x^5 + 4x^4 \\ \hline 6x^3 + 4x^2 - 2x & -12x^4 - 14x^2 \\ & -12x^4 - 8x^3 + 4x^2 \\ \hline 6x^3 + 4x^2 - 4x - 1 & -6x^3 - 4x^2 + 4x + 1 \\ & -6x^3 - 4x^2 + 4x + 1 \end{array}$$

Notice that each trial divisor is equal to the preceding complete divisor, with its last term doubled.

EXAMPLES FOR PRACTICE.

1. $a^4 - 4a^3 + 6a^2 - 4a + 1$ Ans. $a^2 - 2a + 1$
2. $40a + 25 - 14a^2 + 9a^4 - 24a^3$ Ans. $3a^2 - 4a - 5$

THEORY OF EXPONENTS.

25. We have heretofore considered only positive integral exponents. It is often convenient to employ fractional and negative exponents. We defined the positive integral exponent as indicating the number of times a quantity was taken as a factor, thus,

$a^m = a \times a \times a \times a \ldots$. to m factors. We know that $a^n = a \times a \times a \times a \ldots$. to n factors, and we have found, in multiplication, that

$$a^m \times a^n = a^{m+n} \qquad (1)$$

m and n representing any positive integral numbers, for example 4 and 3 giving $a^4 \times a^3 = a^{4+3} = a^7$.

We have further found, under involution, that,

$$(a^m)^n = a^{mn} \qquad (2)$$

We assume that (1) holds true also for fractional and negative exponents.

The meaning of $a^{\frac{1}{3}}$

As (1) holds for all cases

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1$$

or, $a^{\frac{1}{3}}$ is such a quantity that, when raised to the third power, the result is a^1. Therefore, as is seen from evolution, $a^{\frac{1}{3}}$ is the cube root of a^1. In other words $a^{\frac{1}{3}} = \sqrt[3]{a^1}$. We may make this more general by letting m and n represent any positive numbers.

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots\ldots \text{to } n \text{ terms} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}} \ldots\ldots \text{to } n \text{ terms.}$$

$$= a^{\frac{m \times n}{n}} = a^m$$

or, $a^{\frac{m}{n}}$ is a quantity that, when raised to the nth power, gives a^m as a result. Hence $a^{\frac{m}{n}}$ must be the nth root of a^m, or

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

that is, in a fractional exponent, *the numerator denotes a power and the denominator a root*, i. e.,

$$x^{\frac{3}{4}} = \sqrt[4]{x^3}, \quad m^{\frac{1}{5}} = \sqrt[5]{m}, \quad b^{\frac{4}{7}} = \sqrt[7]{b^4}, \text{ etc.}$$

EXAMPLES FOR PRACTICE.

Express the following with radical signs:

1. $a^{\frac{5}{8}}$
2. $x^{\frac{1}{2}}$
3. $m^{\frac{2}{7}}$
4. $y^{\frac{4}{11}}$

Express the following with fractional exponents:

5. $\sqrt{a^3}$
6. $\sqrt[5]{m^6}$
7. $\sqrt[3]{x}$
8. $\sqrt[6]{y^5}$

The value of a numerical quantity with a fractional exponent may be obtained by extracting the root indicated by the denominator, and then raising the root thus obtained to the power indicated by the numerator.

Thus $(-27)^{\frac{2}{3}} = \sqrt[3]{(-27)^2} = (-3)^2 = 9$

EXAMPLES FOR PRACTICE.

1. $(-8)^{\frac{2}{3}}$ Ans. 4.

2. $16^{\frac{5}{4}}$ Ans. 32.

The value of a^0

As (1) must hold universally:

$$a^m \times a^0 = a^{m+0} = a^m$$

but $$a^m \times 1 = a^m$$

therefore $$a^0 = 1$$

That is, *any quantity whose exponent is* 0 *is equal to* 1.

The values of negative exponents.

Take a^{-3} for example.

By (1) $$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1$$

$$\frac{1}{a^3} \times a^3 = \frac{a^3}{a^3} = 1$$

therefore $$a^{-3} = \frac{1}{a^3}$$

In general $$a^{-n} \times a^n = a^{-n+n} = a^0$$

and $$a^{-n} = \frac{1}{a^n}$$

For example,

$$a^{-2} = \frac{1}{a^2}, \quad 7x^{-\frac{1}{2}}y^{-1}z = \frac{7z}{x^{\frac{1}{2}}y}, \text{ etc.}$$

From the preceding may be observed the following:

Any factor of the numerator of a fraction may be transferred to the denominator, and any factor of the denominator to the numerator, the sign of the exponent of that factor being changed at the time of transfer.

Thus the fraction $\frac{a^3b^2}{c^4d}$ may be written in any of the following forms:

$$\frac{a^3b^2c^{-4}}{d}, \quad a^3b^2c^{-4}d^{-1}, \quad \frac{a^3}{b^{-2}c^4d}, \quad \frac{1}{a^{-3}b^{-2}c^4d}, \quad \frac{b^2c^{-4}}{a^{-3}d}, \text{ etc.}$$

EXAMPLES FOR PRACTICE.

Write with positive exponents:

1. x^3y^{-5} 3. $a^{-5}b^{-2}c^3$

2. $4xy^{-\frac{1}{3}}$ 4. $3x^{-\frac{2}{3}}y^{-\frac{3}{7}}$

Transfer the literal factors from the denominators to the numerators in the following:

5. $\dfrac{3}{x^{-2}}$ 6. $\dfrac{5a^2}{2bc^{-3}}$

Transfer the literal factors from the numerators to the denominators in the following:

7. $\dfrac{3x^{\frac{1}{2}}}{4a}$ 8. $\dfrac{4a^{-2}b^{-\frac{2}{3}}}{3c^3}$

As (1) is universal, the following hold good for all values of m and n.

$$a^m \times a^n = a^{m+n}$$

For example,

$$a^3 \times a^{-5} = a^{3-5} = a^{-2}$$

$$a^{\frac{3}{4}} \times a^{-\frac{1}{3}} = a^{\frac{3}{4}-\frac{1}{3}} = a^{\frac{5}{12}}$$

$$a \times a^{-\frac{4}{3}} = a^{1-\frac{4}{3}} = a^{-\frac{1}{3}}$$

and so on.

EXAMPLES FOR PRACTICE.

Find the values of the following:

1. $a^3 \times a^{-2}$
2. $3x \times x^{-\frac{2}{3}}$
3. $2a^{\frac{5}{3}} \times a^{-\frac{2}{3}}$
4. $5c^{-3} \times 3c^{-\frac{1}{2}}$
5. $x^3 y^{-\frac{5}{3}} \times \dfrac{x^{-2} y^{\frac{2}{3}}}{2}$
6. $\dfrac{1}{a^{\frac{1}{2}} b^3} \times \dfrac{3}{a^{-3} b^{\frac{1}{3}}}$

Multiply $x + 2x^{\frac{2}{3}} - 3x^{\frac{1}{3}}$ by $2 - 4x^{-\frac{1}{3}} - 6x^{-\frac{2}{3}}$

$$\begin{array}{l}
x + 2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} \\
2 - 4x^{-\frac{1}{3}} - 6x^{-\frac{2}{3}} \\
\hline
2x + 4x^{\frac{2}{3}} - 6x^{\frac{1}{3}} \\
\quad - 4x^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 12 \\
\qquad\quad - 6x^{\frac{1}{3}} - 12 + 18x^{-\frac{1}{3}} \\
\hline
2x \quad - 20x^{\frac{1}{3}} \quad + 18x^{-\frac{1}{3}}
\end{array}$$ Ans.

NOTE.—In examples like the above, it should be borne in mind that any quantity whose exponent is 0 is equal to 1.

EXAMPLES FOR PRACTICE.

Multiply $a^2 - 2 + a^{-2}$ by $a^2 + 2 + a^{-2}$

Ans. $a^4 - 2 + a^{-4}$

Since the rule for division of exponents holds universally (i. e., $\frac{a^5}{a^3} = a^2$) we now have

$$\frac{a^m}{a^n} = a^{m-n}$$

For example,

$$\frac{a^{-\frac{2}{5}}}{a} = a^{-\frac{2}{5}-1} = a^{-\frac{7}{5}}$$

$$\frac{a^2}{a^{-\frac{1}{2}}} = a^{2+\frac{1}{2}} = a^{\frac{5}{2}}, \text{ etc.}$$

EXAMPLES FOR PRACTICE.

Divide the following:

1. x^5 by x^{-3} 2. a^{-3} by $a^{-\frac{2}{5}}$ Ans. $a^{-\frac{13}{5}}$

Division may be accomplished by the rules previously given.

Divide $x - 3x^{\frac{1}{2}}y^{\frac{1}{2}} + 2y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$

$$\begin{array}{l|l} x - 3x^{\frac{1}{2}}y^{\frac{1}{2}} + 2y & x^{\frac{1}{2}} - y^{\frac{1}{2}} \\ \underline{x - x^{\frac{1}{2}}y^{\frac{1}{2}}} & \overline{x^{\frac{1}{2}} - 2y^{\frac{1}{2}}} \\ -2x^{\frac{1}{2}}y^{\frac{1}{2}} + 2y & \\ \underline{-2x^{\frac{1}{2}}y^{\frac{1}{2}} + 2y} & \\ \end{array}$$

Divide $m - n$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$ Ans. $m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$

26. We will now prove that (2) holds good for all values of m and n.

We will consider three cases in which we may have any value, positive or negative, integral or fractional,

Case I. *Where n is a positive integer.*

By definition,

$$(a^m)^n = a^m \times a^m \times a^m \ldots \text{ to } n \text{ factors.}$$
$$= a^{m+m+m+\ldots} \text{ to } n \text{ terms} = a^{mn}$$

Case II. *Where n is a positive fraction.*

We will denote n by $\frac{p}{q}$

Then $(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$

$$= \sqrt[q]{a^{mp}}$$
$$= a^{\frac{mp}{q}}$$
$$= a^{m \times \frac{p}{q}}$$

Case III. *When n is a negative quantity.*

We will here use $-n$ in place of n.

Then $(a^m)^{-n} = \frac{1}{(a^m)^n}$

$$= \frac{1}{a^{mn}}$$

$$= a^{-mn}$$

$$= a^{m(-n)}$$

Hence $(a^m)^n$ holds for all values of m and n.

For example,

$$(a^{-\frac{2}{5}})^{\frac{1}{2}} = a^{-\frac{2}{5}\times\frac{1}{2}} = a^{-\frac{1}{5}}$$

$$(a^3)^{-2} = a^{3\times-2} = a^{-6}$$

$$(a^{-3})^{-\frac{1}{3}} = a^{-3\times-\frac{1}{3}} = a$$

and so on.

Find the values of the following:

1. $(a^{-2})^3$
2. $(c^{-\frac{2}{5}})^{\frac{10}{3}}$
3. $(a^2)^{-3}$
4. $\frac{1}{(a^2)^{\frac{3}{2}}}$

$(ab)^n = a^n b^n$ for all values of n as may be readily proved.

RADICALS.

27. A **Radical** is a root of a quantity indicated by a radical sign, as $\sqrt{a}$ or $\sqrt[3]{x+y}$.

If the root indicated can be exactly obtained it is called a *rational* quantity; if not, it is an irrational or *surd* quantity.

The degree of the radical is denoted by the index of the radical sign, thus, $\sqrt[3]{a+5}$ is of the third degree.

Similar Radicals are those of the same degree having the same quantity under the radical sign, as $3\sqrt[3]{ab^2}$ and $5\sqrt[3]{ab^2}$.

Most problems in radicals depend on the following law for their solution:

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$$

That is $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

TO REDUCE A RADICAL TO ITS SIMPLEST FORM.

28. A radical is in its simplest form when the quantity is not a perfect power of the degree denoted by any factor of the index of the radical, and has no factor which is a perfect power of the same degree as the radical.

Case I. *When the quantity under the radical sign is a perfect power of the degree denoted by a factor of the index.*

Reduce $\sqrt[6]{81}$ to its simplest form.

$$\sqrt[6]{81} = \sqrt[6]{3^4} = 3^{\frac{4}{6}} = 3^{\frac{2}{3}} = \sqrt[3]{3^2}. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. $\sqrt[3]{8}$.
2. $\sqrt[10]{100}$.
3. $\sqrt[8]{25x^4}$.
4. $\sqrt[6]{125\,a^3\,b^9}$.

Case II. *When the quantity under the radical sign has a factor which is a perfect power of the same degree as the radical.*

Reduce $\sqrt[3]{16}$ to its simplest form.

$$\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2} \quad \text{Ans.}$$

Reduce $\sqrt{9x^5y^5 - 18x^3y^4}$ to its simplest form.

$$\begin{aligned}\sqrt{9x^5y^5 - 18x^3y^4} &= \sqrt{9x^3y^4(y - 2x)} \\ &= \sqrt{9x^3y^4} \times \sqrt{y - 2x} \\ &= 3xy^2\sqrt{y - 2x}\end{aligned}$$

From the preceding examples we derive the following rule:

Resolve the quantity under the radical sign into two factors, one of which is the highest perfect power of the same degree as the radical. Extract the required root of this factor and prefix the result to the indicated root of the other.

NOTE.— If the highest perfect power of the numerical portion is not apparent, that portion should be resolved into its prime factors.

EXAMPLES FOR PRACTICE.

Reduce the following to their simplest forms:

1. $\sqrt{72}$.
2. $\sqrt[4]{80}$.
3. $\sqrt[3]{81a^4b^5}$

4. $\sqrt[3]{63x^4y^5z^6}$

5. $\sqrt{20a^2 + 60a + 45.}$ Ans. $(2a + 3)\sqrt{5}$

6. $\sqrt[3]{54x^4y^5 + 135x^5y^4}$ Ans. $3xy\sqrt[3]{2xy^2 + 5y}$

If the quantity under the radical sign is a fraction, proceed as follows:

Multiply both terms by a quantity that will make the denominator a perfect power of the same degree as the radical. Then proceed as formerly.

Reduce $\sqrt{\frac{9}{8m^3}}$ to its simplest form.

$$\sqrt{\frac{9}{8m^3}} = \sqrt{\frac{18m}{16m^4}} = \sqrt{\frac{9}{16m^4} \times 2m} = \frac{3}{4m^2}\sqrt{2m} \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

Reduce

1. $\sqrt{\frac{3}{2}}$

2. $\sqrt{\frac{4x^3}{27}}$

3. $\sqrt{\frac{xy^2}{4(x + m)}}$ Ans. $\frac{y\sqrt{x^2 + mx}}{2(x + m)}$

Conversely, the coefficient of a radical may be introduced under the radical sign by raising it to the power denoted by the index.

Introduce the coefficient of $2x\sqrt[3]{3y^2}$ under the radical sign.

$$2x\sqrt[3]{3y^2} = \sqrt[3]{8x^3 \times 3y^2} = \sqrt[3]{24x^3y^2} \quad \text{Ans.}$$

NOTE.— A rational quantity may be expressed in the form of a radical by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXAMPLES FOR PRACTICE.

Introduce the coefficients of the following under the radical sign:

1. $3\sqrt[4]{2}$

2. $5x\sqrt[3]{2y^2}$

3. $(a - 1)\sqrt{\frac{a + 1}{a - 1}}$ Ans. $= \sqrt{a^2 - 1}$

ADDITION AND SUBTRACTION OF RADICALS.

29. The sum or difference of two similar radicals is found by prefixing the sum or difference of their coefficients to the *common* radical part.

Find the sum of $\sqrt{27} + \sqrt{12}$

$$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$$

$$\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$$

Hence $\sqrt{27} + \sqrt{12} = 3\sqrt{3} + 2\sqrt{3} = 5\sqrt{3}$

Simplify $\sqrt{\frac{1}{2}} - \sqrt{\frac{9}{8}} + \sqrt{\frac{2}{3}}$

$$\sqrt{\frac{1}{2}} - \sqrt{\frac{9}{8}} + \sqrt{\frac{2}{3}} = \sqrt{\frac{1}{4} \times 2} - \sqrt{\frac{9}{16} \times 2} + \sqrt{\frac{1}{9} \times 6}$$

$$= \frac{1}{2}\sqrt{2} - \frac{3}{4}\sqrt{2} + \frac{1}{3}\sqrt{6}$$

$$= \frac{1}{3}\sqrt{6} - \frac{1}{4}\sqrt{2} \quad \text{Ans.}$$

We may now formulate the following rule:

Reduce each radical to its simplest form. Unite the similar radicals; and indicate the addition or subtraction of the dissimilar.

EXAMPLES FOR PRACTICE.

1. Find the difference between $\sqrt{20}$ and $\sqrt{45}$ Ans. $\sqrt{5}$
2. Between $\sqrt[3]{162}$ and $\sqrt[3]{48}$ Ans. $\sqrt[3]{6}$
3. Find the sum of $\sqrt{128}$, $\sqrt{98}$ and $\sqrt{50}$ Ans. $20\sqrt{2}$

Simplify the following:

4. $\sqrt{\frac{16}{15}} - \sqrt{\frac{3}{5}}$ Ans. $= \sqrt{\frac{1}{15}}$
5. $\sqrt{4x^3y} + \sqrt{9y^3}$ Ans. $(2x + 3y)\sqrt{y}$

To reduce radicals of different degrees to equivalent radicals of the same degree.

30. This may be accomplished by the use of fractional exponents. For example:

Reduce $\sqrt{2}$, $\sqrt[4]{3}$, and $\sqrt[3]{5}$ to equivalent radicals of the same degree.

As we have seen,

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{64}$$

$$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{27}$$

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{4}{12}} = \sqrt[12]{625}$$

Thus our plan is to *write the* radicals with fractional exponents, and reduce these exponents to a common denominator.

NOTE.—The above furnishes a method of comparison between radicals. Thus:

$\sqrt[3]{5} > \sqrt{2} > \sqrt[4]{3}$ because $\sqrt[12]{625} > \sqrt[12]{64} > \sqrt[12]{27}$

EXAMPLES FOR PRACTICE.

Reduce the following to equivalent radicals of the same degree:

1. $\sqrt[3]{2}$, $\sqrt[3]{4}$, $\sqrt{5}$ Ans. $\sqrt[6]{4}$, $\sqrt[6]{16}$, $\sqrt[6]{125}$
2. $\sqrt[3]{2x}$, $\sqrt[5]{3y}$, $\sqrt[5]{4z}$
   Ans. $\sqrt[15]{32x^5}$, $\sqrt[15]{27y^3}$, $\sqrt[15]{64z^3}$
3. Which is the greater, $\sqrt[4]{3}$ or $\sqrt[5]{5}$? Ans. $\sqrt[5]{5}$

MULTIPLICATION OF RADICALS.

31. Multiply $\sqrt{6}$ by $\sqrt{30}$

We have already seen that $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

Hence $\sqrt{6} \times \sqrt{30} = \sqrt{6 \times 30} = \sqrt{180} = 6\sqrt{5}$

Multiply $\sqrt{2x}$ by $\sqrt[3]{3x^2}$

Reducing to equivalent radicals of the same degree,

$$\sqrt{2x} = (2x)^{\frac{1}{2}} = (2x)^{\frac{3}{6}} = \sqrt[6]{(2x)^3} = \sqrt[6]{8x^3}$$

$$\sqrt[3]{3x^2} = (3x^2)^{\frac{1}{3}} = (3x^2)^{\frac{2}{6}} = \sqrt[6]{(3x^2)^2} = \sqrt[6]{9x^4}$$

Hence $\sqrt{2x} \times \sqrt[3]{3x^2} = \sqrt[6]{8x^3} \times \sqrt[6]{9x^4} = \sqrt[6]{72x^7}$

$$= x\sqrt[6]{72x} \quad \text{Ans.}$$

From the above, we derive the following rule:

Reduce the radicals to equivalent radicals of the same degree, find the product of the quantities under the radical signs, and write that product under the common radical sign. Reduce to simplest form.

ILLUSTRATIVE EXAMPLES.

$2\sqrt{3} + 3\sqrt{2}$ by $3\sqrt{3} - \sqrt{2}$

$$\begin{array}{r} 2\sqrt{3} + 3\sqrt{2} \\ 3\sqrt{3} - \sqrt{2} \\ \hline 18 + 9\sqrt{6} \\ \quad - 2\sqrt{6} - 6 \\ \hline 18 + 7\sqrt{6} - 6 = 12 + 7\sqrt{6} \quad \text{Ans.} \end{array}$$

NOTE.— Multiplying a radical of the second degree by itself merely removes the radical sign, as $\sqrt{3} \times \sqrt{3} = 3$

$3\sqrt{x^2 + 1} + 4x$ by $\sqrt{2x^2 + 1} - x$

$$\begin{array}{r} 3\sqrt{x^2 + 1} + 4x \\ 2\sqrt{x^2 + 1} - x \\ \hline 6(x^2 + 1) + 8x\sqrt{x^2 + 1} \\ \quad - 3x\sqrt{x^2 + 1} - 4x^2 \\ \hline 6x^2 + 6 + 5x\sqrt{x^2 + 1} - 4x^2 \\ = 2x^2 + 6 + 5x\sqrt{x^2 + 1} \quad \text{Ans.} \end{array}$$

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $2\sqrt{3x}$ by $5\sqrt{15x}$ Ans. $30x\sqrt{5}$
2. $\sqrt[3]{4x^2}$ by $\sqrt{2x}$ Ans. $2x\sqrt[6]{2x}$
3. $\sqrt{x} - 2$ by $\sqrt{x} + 3$ Ans. $x + \sqrt{x} - 6$
4. $\sqrt{a} - \sqrt{b} + \sqrt{c}$ by $\sqrt{a} + \sqrt{b} - \sqrt{c}$
   Ans. $a - b + 2\sqrt{bc} - c$
5. $\sqrt{a + 1} - 2\sqrt{a}$ by $2\sqrt{a + 1} + \sqrt{a}$
   Ans. $2 - 3\sqrt{a^2 + a}$

Expand the following:

6. $(2\sqrt{3} - 3)^2$ Ans. $21 - 12\sqrt{3}$
7. $(\sqrt{m^2 + 1} + m) \times (\sqrt{m^2 + 1} - m)$ Ans. 1

DIVISION OF RADICALS.

32. Again applying the principle that

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$$

we see that $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{b}} = \sqrt[n]{a}$, and derive the following rule:

Reduce the radicals to equivalent radicals of the same degree, find the quotient obtainable from the division of one quantity under the radical sign by the other, and write that quotient under the common radical sign.

Divide $\sqrt[3]{15}$ by $\sqrt{5}$

Reducing to equivalent radicals of the same degree,

$$\frac{\sqrt[3]{15}}{\sqrt{5}} = \frac{\sqrt[6]{225}}{\sqrt[6]{125}} = \sqrt[6]{\frac{225}{125}} = \sqrt[6]{\frac{9}{5}} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Divide

1. $\sqrt{6}$ by $\sqrt[3]{3}$ Ans. $\sqrt[6]{24}$
2. $\sqrt[3]{4m}$ by $\sqrt[4]{2m}$ Ans. $\sqrt[12]{32m}$
3. $\sqrt[3]{3x^2y}$ by $\sqrt[5]{6x^3y^3}$ Ans. $\sqrt[15]{\frac{9x}{8y}}$

INVOLUTION AND EVOLUTION OF RADICALS.

33. Any power or root of a radical may be found by using fractional exponents.

Raise $\sqrt[6]{12}$ to a third power.

$(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}} = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3}$ Ans.

Raise $\sqrt[3]{2}$ to the fifth power.

$(\sqrt[3]{2})^5 = (2^{\frac{1}{3}})^5 = 2^{\frac{5}{3}} = \sqrt[3]{2^5} = \sqrt[3]{32} = 2\sqrt[3]{4}$ Ans.

From the above we derive the following rule:

When possible divide the index by the exponent of the required power; in other cases raise the quantity under the radical sign to the required power.

Thus, $(\sqrt[6]{12})^3 = \sqrt{12} = 2\sqrt{3}$. The second example illustrates the second case.

EXAMPLES FOR PRACTICE.

Find the values of the following:

1. $(\sqrt[4]{7})^2$ Ans. $\sqrt{7}$
2. $(\sqrt[3]{x-y})^4$ Ans. $\sqrt[4]{x-y}$
3. $(\sqrt[12]{32})^3$ Ans. $2\sqrt[4]{2}$
4. $(\sqrt[3]{3abc})^4$ Ans. $3abc\sqrt[3]{3abc}$

Find the square root of $\sqrt[3]{6}$

$$\sqrt{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{2}} = 6^{\frac{1}{6}} = \sqrt[6]{6} \quad \text{Ans.}$$

Find the cube root of $\sqrt{27a^3}$

$$\sqrt[3]{(\sqrt{27a^3})} = (\sqrt{27a^3})^{\frac{1}{3}} = (\sqrt{(3a)^3})^{\frac{1}{3}} = [(3a)^{\frac{3}{2}}]^{\frac{1}{3}}$$

$$= (3a)^{\frac{1}{2}} = \sqrt{3a} \quad \text{Ans.}$$

From the preceding, the following rule is evident:

When possible, extract the required root of the quantity under the radical sign; in other cases multiply the index of the radical by the index of the required root.

Should the radical have a coefficient, not a perfect power, of the same degree as the root required, introduce that coefficient under the radical sign before applying the rule.

Thus,

$$\sqrt[3]{2\sqrt{2}} = \sqrt[3]{\sqrt{8}} = \sqrt{2}$$

Find the values of the following:

1. $\sqrt[3]{3\sqrt{3}}$ Ans. $\sqrt{3}$

2. $\sqrt{(\sqrt[3]{a^2} - 2a + 1)}$ Ans. $\sqrt[3]{a} - 1$

To reduce a fraction having an irrational denominator to an equivalent fraction with a rational denominator.

34. **Case I.** *When the denominator is a monomial.*

Reduction is accomplished by multiplying both terms by a radical of the same degree as the denominator, having a quantity under the radical sign, that will make the denominator of the resulting fraction rational. To illustrate,

Reduce $\frac{5}{\sqrt[3]{3x^2}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by the cube root of $9x$ we have

$$\frac{5\sqrt[3]{9x}}{\sqrt[3]{3x^2}\sqrt[3]{9x}} = \frac{5\sqrt[3]{9x}}{\sqrt[3]{27x^3}} = \frac{5\sqrt[3]{9x}}{3x} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Reduce the following to equivalent fractions with rational denominators:

1. $\frac{3}{\sqrt{2}}$ Ans. $\frac{3\sqrt{2}}{2}$

Case II. *When the denominator is a binomial containing radicals of the second degree only.*

Reduce $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ to an equivalent.

Multiplying both terms by $\sqrt{5}-\sqrt{2}$, we have

$$\frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})} = \frac{5-2\sqrt{10}+2}{5-2}$$

$$= \frac{7-2\sqrt{10}}{3} \quad \text{Ans.}$$

Reduce $\frac{1+\sqrt{1-a}}{1-\sqrt{1-a}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $1+\sqrt{1-a}$, we have

$$\frac{(1+\sqrt{1-a})^2}{(1-\sqrt{1-a})(1+\sqrt{1-a})} = \frac{1+2\sqrt{1-a}+1-a}{1-(1-a)}$$

$$= \frac{2-a+2\sqrt{1-a}}{a} \quad \text{Ans.}$$

We now come to the rule:

Multiply both terms of the fraction by the denominator with the sign between its terms changed.

EXAMPLES FOR PRACTICE.

Reduce the following to equivalent fractions having rational denominators.

1. $\frac{4-\sqrt{3}}{2-\sqrt{3}}$ Ans. $5+2\sqrt{3}$.

2. $\frac{m-\sqrt{n}}{m+\sqrt{n}}$ Ans. $\frac{m^2-2m\sqrt{n}+n}{m^2-n}$

3. $\frac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$ Ans. $\frac{a+\sqrt{a^2-b^2}}{b}$

APPROXIMATE VALUE OF A FRACTION WITH AN IRRATIONAL DENOMINATOR.

35. This is best determined by reducing it to an equivalent fraction with a rational denominator.

Find the approximate value of $\frac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}} = \frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2+\sqrt{2}}{4-2}$$
$$= \frac{2+1.414\ \ldots}{2} = 1.707\ \ldots\ldots \text{ Ans.}$$

EXAMPLE FOR PRACTICE.

1. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ Ans. .101 . . .

IMAGINARY QUANTITIES.

36. An **imaginary quantity** is an indicated even root of a negative quantity, as $\sqrt{-3}$, or $\sqrt[4]{-x^2}$. On the other hand, distinct from these, all other quantities, rational or irrational, are called *real* quantities.

Every imaginary square root can be expressed as the product of a *real* quantity, multiplied by $\sqrt{-1}$. For example,

$\sqrt{-x^2} = \sqrt{x^2 \times (-1)} = \sqrt{x^2} \times \sqrt{-1} = x\sqrt{-1}$
$\sqrt{-7} = \sqrt{7 \times (-1)} = \sqrt{7}\sqrt{-1}$; etc.

Suppose now that we find the powers of $\sqrt{-1}$. $\sqrt{-1}$ designates a quantity which when multiplied by itself, produces -1.

Hence,

$$(\sqrt{-1})^2 = -1.$$

Therefore,

$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = (-1) \times \sqrt{-1} = -\sqrt{-1}$
$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1$
$(\sqrt{-1})^5 = (\sqrt{-1})^4 \times \sqrt{-1} = 1 \times \sqrt{-1} = -\sqrt{-1}$
and so on.

From the preceding we perceive that the first four powers of $\sqrt{-1}$ are

$$\sqrt{-1},\ -1,\ -\sqrt{-1},\ \text{and}\ 1$$

and for the higher powers these terms occur over and over again, and in the same order.

MULTIPLICATION AND DIVISION OF IMAGINARY QUANTITIES.

The product of two or more imaginary square roots may be found in the following manner:

Multiply the $\sqrt{-2}$ by $\sqrt{-3}$.

$$\sqrt{-2} \times \sqrt{-3} = \sqrt{2}\sqrt{-1} \times \sqrt{3}\sqrt{-1}$$
$$= \sqrt{2}\sqrt{3}\,(\sqrt{-1})^2$$
$$= \sqrt{6} \times (-1) = -\sqrt{6} \qquad \text{Ans.}$$

Multiply $\sqrt{-x^2}$, $\sqrt{-y^2}$, and $\sqrt{-z^2}$

$$\sqrt{-x^2} \times \sqrt{-y^2} \times \sqrt{-z^2} = x\sqrt{-1} \times y\sqrt{-1} \times z\sqrt{-1}$$
$$= xyz\,(\sqrt{-1})^3 = -xyz\sqrt{-1} \quad \text{Ans.}$$

From the above we derive the following rule:

Reduce each imaginary quantity to the form of a real quantity multiplied by $\sqrt{-1}$. *Form the product of the real quantities, and multiply the result by the required power of the* $\sqrt{-1}$.

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $4\sqrt{-3}$ and $2\sqrt{-2}$ — Ans. $-8\sqrt{6}$
2. $-3\sqrt{-a}$ and $4\sqrt{-b}$ — Ans. $12\sqrt{ab}$
3. $\sqrt{-3}$, $\sqrt{-4}$, and $\sqrt{-5}$ — Ans. $-2\sqrt{-15}$
4. $\sqrt{-m^2}$ and $\sqrt{-n^2}$ — Ans. $-mn$
5. $1 - 2\sqrt{-1}$ and $3 + \sqrt{-1}$ — Ans. $5 - \sqrt{-1}$

Expand the following:

6. $(2 - \sqrt{-3})^2$ — Ans. $1 - 4\sqrt{-3}$
7. $(x + \sqrt{-y})\,(x - \sqrt{-y})$ — Ans. $x^2 + y$
8. $(1 + \sqrt{-1})^2 + (1 - \sqrt{-1})^2$ — Ans. 0

Divide $\sqrt{-a}$ by $\sqrt{-b}$

$$\frac{\sqrt{-a}}{\sqrt{-b}} = \frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}\sqrt{-1}} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

EXAMPLES FOR PRACTICE.

Divide the following:

1. $\sqrt{-24}$ by $\sqrt{-3}$ Ans. $2\sqrt{2}$
2. $\sqrt[6]{-54}$ $\sqrt[6]{-2}$ Ans. $\sqrt[6]{3}$

PROPERTIES OF QUADRATIC SURDS.

37. A **quadratic surd** is the indicated square root of an imperfect square, as $\sqrt{5}$ or $\sqrt{13}$.

The following laws are of assistance in dealing with these quantities.

I. A quadratic surd cannot be equal to a rational quantity plus a quadratic surd.

To show this, let the following equation be true, if possible:

$$\sqrt{a} = b + \sqrt{c}$$

Squaring the equation, $a = b^2 + 2b\sqrt{c} + c$

or $$2b\sqrt{c} = a - b^2 - c$$

whence $$\sqrt{c} = \frac{a - b^2 - c}{2b}$$

That is, a surd is equal to a rational quantity, which is impossible.

Hence, $\sqrt{a}$ cannot be equal to $b + \sqrt{c}$

II. If $a + \sqrt{b} = c + \sqrt{d}$, then

$$a = c \text{ and } b = d$$

If a is not equal to c, let

$$a = c + x$$

Substituting we have

$$c + x + \sqrt{b} = c + \sqrt{d}$$

or $x + \sqrt{b} = \sqrt{d}$, which is impossible by I;

Hence $a = c$ and consequently $\sqrt{b} = \sqrt{d}$

III. If $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$$

Squaring the equation $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

we have $$a + \sqrt{b} = x + 2\sqrt{xy} + y$$

whence by I, $$a = x + y, \qquad (1)$$

and $$\sqrt{b} = 2\sqrt{xy} \qquad (2)$$

Subtracting (2) from (1),

$$a - \sqrt{b} = x - 2\sqrt{xy} + y,$$

Extracting square root,

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$$

SQUARE ROOT OF A BINOMIAL SURD.

38. The preceding principles apply when extracting the square root of a binomial surd whose first term is rational.

For example, required the square root of $13 - \sqrt{160}$

Assume	$\sqrt{13 - \sqrt{160}} = \sqrt{x} - \sqrt{y}$	(1)
Then, by Art. 37, III,	$\sqrt{13 + \sqrt{160}} = \sqrt{x} + \sqrt{y}$	(2)
Multiplying (1) by (2),	$\sqrt{169 - 160} = x - y$	
Or,	$x - y = 3$	(3)
Squaring (1),	$13 - \sqrt{160} = x - \sqrt{2xy} + y$	
Whence, by Art. 37, II,	$x + y = 13$	(4)
Adding (3) and (4),	$2x = 16$, or $x = 8$	
Subtracting (3) from (4),	$2y = 10$, or $y = 5$	
Substituting in (1),	$\sqrt{13 - \sqrt{160}} = \sqrt{8} - \sqrt{5}$	
	$= 2\sqrt{2} - 5$	Ans.

Examples like the above may often be solved by inspection by expressing the given quantity in the form of a perfect trinomial square, as follows:

Reduce the surd term so that its coefficient may be 2. Separate the rational term into two parts, whose product is the quantity under the radical sign. Extract the square root of these parts, and connect them by the sign of the surd term.

Extract the square root of $8 + \sqrt{48}$

$$\sqrt{8 + \sqrt{48}} = \sqrt{8 + 2\sqrt{12}}$$

We then separate 8 into two parts whose product is 12. These parts are 6 and 2; hence,

$$\sqrt{8 + 2\sqrt{12}} = \sqrt{6 + 2\sqrt{6 \times 2} + 2}$$
$$= \sqrt{6} + \sqrt{2} \quad \text{Ans.}$$

Extract the square root of $22 - 3\sqrt{32}$

$$\sqrt{22 - 3\sqrt{32}} = \sqrt{22 - \sqrt{288}} = \sqrt{22 - 2\sqrt{72}}$$

We then separate 22 into two parts whose product is 72. The parts are 18 and 4; hence,

$$\sqrt{22 - 3\sqrt{32}} = \sqrt{18 - 2\sqrt{72} + 4}$$
$$= \sqrt{18} - \sqrt{4} = 3\sqrt{2} - 2 \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Extract the square roots of the following:

1. $9 - 4\sqrt{5}$ Ans. $\sqrt{5} - 2$
2. $20 - 5\sqrt{12}$ Ans. $\sqrt{15} - \sqrt{5}$
3. $67 - 7\sqrt{72}$ Ans. $7 - 3\sqrt{2}$

SOLUTION OF EQUATIONS CONTAINING RADICALS.

39. Solve the equation $\sqrt{x^2 - 5} - x = -1$

Transposing, $\sqrt{x^2 - 5} = x - 1$

Squaring both members, $x^2 - 5 = x^2 - 2x + 1$

Transposing and uniting terms, $2x = 6$

$x = 3$ Ans.

Solve the equation $\sqrt{2x - 1} + \sqrt{2x + 6} = 7$

Transposing $\sqrt{2x - 1}$

$$\sqrt{2x + 6} = 7 - \sqrt{2x - 1}$$

Squaring, $2x + 6 = 49 - 14\sqrt{2x - 1} + 2x - 1$

$$14\sqrt{2x - 1} = 42$$

or $\sqrt{2x - 1} = 3$

squaring $2x - 1 = 9$

$2x = 10$

$x = 5$ Ans.

From the above we derive the following rule:

Transpose the terms of the equation so that a radical term may stand alone in one member; then raise both members to a power of the same degree as the radical. If there are still radical terms remaining, repeat the operation.

EXAMPLES FOR PRACTICE.

Solve the following equations:

1. $\sqrt{5a-1}-2=1$ Ans. $a=2$
2. $\sqrt[3]{4m+3}=3$ Ans. $m=6$
3. $\sqrt{x-32}+\sqrt{x}=16$ Ans. $x=81$
4. $\sqrt{3x+10}-\sqrt{3x+25}=-3$ Ans. $x=-3$

QUADRATIC EQUATIONS.

40. A **quadratic equation** or an equation of the second degree, is one in which the *square* is the highest power of the unknown quantity. Such an equation is termed *pure* when it contains only the square of the unknown quantity; as $ax^2=b$. It is termed *affected* when it contains both the square and the first power of the unknown quantity; as $ax^2+bx+c=0$.

PURE QUADRATIC EQUATIONS.

41. A pure quadratic equation is solved by reducing it to the form $x^2=a$, and then extracting the square roots of both members.

ILLUSTRATIVE EXAMPLES.

Solve the equation $3x^2+7=\frac{5x^2}{4}+35$

Clearing of fractions, $12x^2+28=5x^2+140$

Transposing and uniting, $7x^2=112$

$x^2=16$

Taking the square root of both, $x=\pm 4$ Ans.

NOTE. *We place the double sign $\pm$ before the result, as it may be either positive or negative.*

Solve the equation $7x^2-5=5x^2-13$

Transposing and uniting, $2x^2=-8$

$x^2=-4$

$x=\pm\sqrt{-4}$

$=\pm 2\sqrt{-1}$

EXAMPLES FOR PRACTICE.

1. Solve the equation

$$4x^2 - 7 = 29 \qquad \text{Ans. } x = \pm 3$$

2. Solve the equation

$$\frac{5}{6x^2} - \frac{7}{4x^2} = -\frac{33}{16} \qquad \text{Ans. } x = \pm \frac{2}{3}$$

3. Solve the equation:

$$(3x - 2)(2x + 5) + (5x + 1)(4x - 3) - 91 = 0$$

$$\text{Ans. } x = \pm 2$$

AFFECTED QUADRATIC EQUATIONS.

42. To solve an affected quadratic equation, add to both members such a quantity as will render the first member a perfect square, an operation which is termed "completing the square."

There are two methods; the first is as follows:

Reduce the equation to the form $x^2 + px = q$ (*p and q representing any quantities whatever, positive or negative, integral or fractional.*)

Complete the square by adding to both members the square of half the coefficient of x.

Extract the square root of both members, and solve the simple equation thus formed.

For example, let it be required to solve the equation $x^2 + 3x = 4$

We have seen that in any trinomial square, the middle term is twice the product of the square roots of the first and third terms; hence the square root of the third term is equal to *the second term divided by twice the square root of the first.*

Therefore the *square root* of the quantity which must be added to $x^2 + 3x$ to "complete the square," is $\frac{3x}{2x}$ or $\frac{3}{2}$ Adding the *square* of the latter to both members, we have

$$x^2 + 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}$$

$$x + \frac{3}{2} = \pm \frac{5}{2}$$

Transposing $\frac{3}{2}$, $\qquad x = -\frac{3}{2} + \frac{5}{2}$, or $-\frac{3}{2} - \frac{5}{2}$

Whence, $\qquad x = 1$, or -4 $\qquad$ Ans.

ILLUSTRATIVE EXAMPLE.

Solve the equation $3x^2 - 8x = -4$

To obtain for the equation the form, $x^2 + px = q$, we divide by 3, and have

$$x^2 - \frac{8x}{3} = -\frac{4}{3}$$

Adding to both members the square of $\frac{4}{3}$,

$$x^2 - \frac{8x}{3} + \frac{16}{9} = -\frac{4}{3} + \frac{16}{9} = \frac{4}{9}$$

Extracting the square root,

$$x - \frac{4}{3} = \pm\frac{2}{3}$$

Whence, $x = 2$, or $\frac{2}{3}$ Ans.

NOTE. To verify these values, put $x = 2$ into the given equation, $12 - 16 = -4$; put $x = \frac{2}{3}$, $\frac{4}{3} - \frac{16}{3} = -4$

Solve $-3x^2 - 7x = \frac{10}{3}$

(When the coefficient of x^2 is negative, all the signs must be changed.)

Dividing by -3, $x^2 + \frac{7x}{3} = -\frac{10}{9}$

adding to both members the square of $\frac{7}{6}$,

$$x^2 + \frac{7x}{3} + \frac{49}{36} = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}$$

extracting square root, $x + \frac{7}{6} = \pm\frac{3}{6}$

whence, $x = -\frac{2}{3}$, or $-\frac{5}{3}$ Ans.

EXAMPLES FOR PRACTICE.

Solve the equations:

1. $x^2 - 5x = -4$ Ans. $x = 4$ or 1
2. $x^2 + x = 6$ Ans. $x = 2$ or -3

ILLUSTRATIVE EXAMPLE.

Solve $9x^2 - 5x = 4$

(If the coefficient of x^2 is a perfect square, it simplifies matters to merely add to both members the square of the quotient obtained by dividing the coefficient of the second term by twice the square root of the coefficient of the first,) as follows:

Adding the square of $\frac{5}{6}$, the result obtained by the above process, to both members,

$$9x^2 - 5x + \frac{25}{36} = 4 + \frac{25}{36} = \frac{169}{36}$$

extracting square root,

$$3x - \frac{5}{6} = \pm\frac{13}{6}$$

$$3x = \frac{5}{6} \pm \frac{13}{6} = 3 \text{ or } -\frac{4}{3}$$

whence $$x = 1, \text{ or } -\frac{4}{9}$$

NOTE.—If the coefficient of x^2 is not a perfect square, it may be made so by multiplication, i. e., multiplying all the terms by the same number; as $18x^2 + 5x = 2$ multiplied by 2, gives, $36x^2 + 10x = 4$.

EXAMPLES FOR PRACTICE.

1. Solve $25x^2 - 15x = -2$ Ans. $x = \frac{2}{5}$ or $\frac{1}{5}$

2. Solve $8x^2 + x - 34 = 0$ Ans. $x = 2$ or $-\frac{17}{8}$

Second method of completing the square.

Reduce the equation to the form, $ax^2 + bx = c$

NOTE. The advantage of this method over the preceding is that fractions are avoided in completing the square.

Multiply both members by 4 times the coefficient of x^2, and add to each the square of the coefficient of x, in the given equation.

Extract the square root of both members, and solve the simple equation thus formed.

For example, multiplying each term of $ax^2 + bx = c$ by $4a$, we have

$$4a^2x^2 + 4abx = 4ac$$

Completing the square by adding square of b

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac$$

Extracting square root

$$2ax + b = \pm\sqrt{b^2 + 4ac}$$

$$2ax = -b \pm \sqrt{b^2 + 4ac}$$

Dividing by $2a$,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

ILLUSTRATIVE EXAMPLE.

Solve the equation $2x^2 - 7x = -3$

Multiplying both members by 4 times 2 or 8,

$$16x^2 - 56x = -24$$

Adding to each member the square of 7,

$$16x^2 - 56x + 49 = -24 + 49 = 25$$

Extracting square roots,

$$4x - 7 = \pm 5$$

Transposing,

$$4x = 7 \pm 5$$

$$x = 3 \text{ or } \frac{1}{2} \qquad \text{Ans.}$$

This process may be shortened, if the coefficient of x in the given equation is an even number, fractions being avoided, and the rule modified as follows:

Multiply both the members by the coefficient of x^2, and add to each the square of half the coefficient of x.

Solve $7x^2 + 4x = 51$

Multiplying both members by 7,

$$49x^2 + 28x + 4 = 361$$

Extracting the square root,

$$7x + 2 = \pm 19$$

Whence, $7x = -2 \pm 19$

$$x = \frac{17}{7} \text{ or } -3 \qquad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. Solve $4x^2 - x = 3$ Ans. $x = 1$ or $-\frac{3}{4}$

2. Solve $2x^2 - 7x = 15$ Ans. $x = 5$ or $-\frac{3}{2}$

3. Solve $7x^2 - 16x + 4 = 0$ Ans. $x = 2$ or $\frac{2}{7}$

The same methods are used in solving literal quadratic equations.

ILLUSTRATIVE EXAMPLES.

Solve $x^2 - 2mx = 2m + 1$

Completing the square by adding m^2 to both members,

$x^2 - 2mx + m^2 = m^2 + 2m + 1 = (m + 1)^2$

Extracting the square root,

$$x - m = \pm (m + 1)$$

Whence,

$$x = m + (m + 1) \text{ or } m - (m + 1)$$

$$x = 2m + 1 \text{ or } -1 \qquad \text{Ans.}$$

Solve $x^2 + ax - bx - ab = 0$

Write the equation, $x^2 + (a - b)\, x = ab$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2 + 4\,(a - b)\, x = 4\,ab$$

Adding to each member the square of $a - b$

$$4x^2 + 4\,(a - b)\, x + (a - b)^2 = (a - b)^2 + 4ab$$

$$= a^2 + 2ab + b^2$$

Extracting the square root,

$$2x + (a - b) = \pm (a + b)$$

Whence,

$$2x = -(a - b) \pm (a + b)$$

$$2x = -a + b + a + b = 2b$$

$$2x = -a + b - a - b = -2a$$

$$x = -a \text{ or } b \qquad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Solve the equation,

1. $x^2 - 2ax = (b + a)\,(b - a)$

Ans. $x = (a + b)$ or $(a - b)$

2. $x^2 - m^2\,(1 - m)\, x = m^3$ Ans. $x = m^2$ or $-m^2$

We have shown above, (second method of completing the square,) that if $ax^2 + bx = c$

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

This result may be used as a formula for the solution of "quadratics," as follows:

ILLUSTRATIVE EXAMPLES.

Solve $3x^2 + 5x = 12$

In this case $a = 3$, $b = 5$, and $c = 12$

Substituting these values in the formula

$$x = \frac{-5 \pm \sqrt{25 + 144}}{6} = \frac{-5 \pm \sqrt{169}}{6}$$

$$= \frac{-5 \pm 13}{6} = \frac{3}{4} \text{ or } -3 \qquad \text{Ans.}$$

Solve $110x^2 - 21x = -1$

In this case, $a = 110$, $b = -21$, $c = -1$; therefore,

$$x = \frac{21 \pm \sqrt{441 - 440}}{220} = \frac{21 \pm 1}{220} = \frac{1}{10} \text{ or } \frac{1}{11} \quad \text{Ans.}$$

1. Solve by this method,

$2x^2 + 5x = 18$ Ans. $x = 2$ or $-\frac{9}{2}$

2. Solve by this method,

$27 + 39x - 10x^2 = 0$ Ans. $x = \frac{9}{2}$ or $-\frac{3}{5}$

PROBLEMS.

43. Involving quadratic equations.

ILLUSTRATIVE EXAMPLES.

A man sold a watch for \$21.00 and lost as many per cent as the watch cost him in dollars. Required the cost of the watch.

Let x = the cost in dollars.

Then x = the loss per cent,

and x times $\frac{x}{100} = \frac{x^2}{100}$ = the loss in dollars.

By the conditions $\frac{x^2}{100} = x - 21$

Solving this equation, $x = 70$ or 30.

That is, the watch cost either \$70 or \$30, for each of these values satisfies the given conditions.

A farmer bought some sheep for \$72. If he had bought six more for the same money they would have cost \$1 apiece less. How many did he buy?

Let x = the number bought.

Then $\frac{72}{x}$ = the price paid for one,

and $\frac{72}{x+6}$ = the price if there had been six more.

By the conditions, $\frac{72}{x} = \frac{72}{x+6} + 1$

Solving $x = 18$ or -24

The answer is 18, because the negative value, — 24, does not answer the conditions of the problem. This principle applies to all quadratics.

EXAMPLES FOR PRACTICE.

1. I bought a lot of flour for \$175; and the number of dollars per barrel was $\frac{7}{4}$ of the number of barrels. How many barrels were purchased, and at what price?

Ans. 10 bbls. at \$17.50.

2. A certain company agreed to build a vessel for \$6,300; but, two of their number having died, the rest had each to advance \$200 more than they otherwise would have done. Of how many persons did the company consist at first? Ans. 9.

3. A grazier bought a certain number of oxen for \$240. Having lost 3, he sold the remainder at \$8 a head more than they cost him, and gained \$59. How many did he buy? Ans. 16.

4. A certain number consists of two digits, the left-hand digit being twice the right hand. If the digits are reversed, the product of the number thus formed, increased by 11, and the original number, is 4,956. Find the number. Ans. 84.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

44. To determine the degree of an equation containing more than one unknown quantity, observe the greatest sum of the exponents of the unknown quantities in any term. For

example, $2x + 3xy = 4$ is an equation of the second degree, (We assume that the equation has been cleared of fractions, and freed from radical signs and fractional and negative exponents.)

When we combine, by elimination, two equations of the second degree, containing two unknown quantities, we generally obtain an equation of the fourth degree with one unknown quantity.

It is evident therefore that the process usually employed in quadratics will not be sufficient to solve all simultaneous equations of the second degree.

There are, however, three cases in which the solution may be effected by the ordinary rules, as follows:

Case I. *When one equation is of the first degree.*

Case II. *When the equations are symmetrical with respect to x and y.*

Case III. *When each equation is of the second degree and homogeneous, (i. e. when all its terms are homogeneous.)*

EXAMPLES TO ILLUSTRATE EACH CASE.

Case I. In this case the equations may be solved by finding the value of one of the unknown quantities in terms of the other from the simple equation, and then substituting.

Solve the equations.

$$\begin{cases} 2x^2 - xy = 6y & (1) \\ x + 2y = 7 & (2) \end{cases}$$

From (2), $2y = 7 - x$, or $y = \dfrac{7-x}{2}$ (3)

Substituting in (1), $2x^2 - x\left(\dfrac{7-x}{2}\right) = 6\left(\dfrac{7-x}{2}\right)$

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$

$$5x^2 - x = 42$$

Solving this equation, $x = 3$ or $-\dfrac{14}{5}$

Substituting in (3), $y = \dfrac{7-3}{2}$ or $\dfrac{7+\frac{14}{5}}{2}$

$$= 2 \text{ or } \frac{49}{10}$$

Ans. $x = 3, y = 2$; or $x = -\dfrac{14}{5}, y = \dfrac{49}{10}$

EXAMPLES FOR PRACTICE.

1. Solve the equations,

$$\left.\begin{aligned} 2x^2 - 3y^2 &= -10 \\ 3x + y &= 1 \end{aligned}\right\} \quad \text{Ans.} \quad \begin{cases} x = 1 \text{ or } -\frac{7}{25} \\ y = -2 \text{ or } \frac{46}{25} \end{cases}$$

2. Solve the equations,

$$\left.\begin{aligned} 3x^2 - 2xy &= 15 \\ 2x + 3y &= 12 \end{aligned}\right\} \quad \text{Ans.} \quad \begin{cases} x = 3 \text{ or } -\frac{15}{13} \\ y = 2 \text{ or } \frac{62}{13} \end{cases}$$

Case II. In this case, the equations must be combined in such a way as to give the values of the sum and difference of the unknown quantities.

NOTE. For an equation to be symmetrical with respect to two quantities, it must be possible to interchange them, without destroying the equality. Thus $x^2 + xy + y^2 = 1$ is symmetrical with respect to x and y, while $x - y = 1$ obviously is not.

Solve the equations,

$$\begin{cases} x + y = 2 & (1) \\ xy = -15 & (2) \end{cases}$$

Squaring, (1), $x^2 + 2xy + y^2 = 4$ (3)

Multiplying, (2) by 4, $4xy = -60$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 64$

Extracting square root, $x - y = \pm 8$ (5)

Adding (1) and (5), $2x = 2 \pm 8 = 10 \text{ or } -6$

Whence, $x = 5 \text{ or } -3$

Subtracting (5) from (1), $2y = 2 \mp 8 = -6 \text{ or } 10$

Whence, $y = -3 \text{ or } 5$

Ans. $x = 5, y = -3$; or $x = -3, y = 5$

NOTE. These equations could have been solved as in Case I, but the symmetrical method is simpler.

Solve the equations,

$$\begin{cases} x^2 + y^2 = 50 & (1) \\ x - y = 8 & (2) \end{cases}$$

Squaring (2), $x^2 - 2xy + y^2 = 64$ (3)

Subtracting (3) from (1), $2xy = -14$ (4)

Adding (1) and (4), $x^2 + 2xy + y^2 = 36$

Whence, $x + y = \pm 6$ (5)

Adding (2) and (5), $2x = 8 \pm 6 = 14$ or 2

Whence, $x = 7$ or 1

Subtracting (2) from (5), $2y = -8 \pm 6 = -2$ or -14

Whence, $y = -1$ or -7

Ans. $x = 7, y = -1$; or $x = 1, y = -7$

Solve the equations,

$$\begin{cases} x^2 + y^2 = 133 & (1) \\ x^2 - xy + y^2 = 19 & (2) \end{cases}$$

Dividing (1) by (2), $x + y = 7$ (3)

Squaring, $x^2 + 2xy + y^2 = 49$ (4)

Subtracting (4) from (2), $3xy = -30$

Whence, $xy = -10$ (5)

Adding (2) and (5), $x^2 - 2xy + y^2 = 9$

Whence, $x - y = \pm 3$ (6)

Adding (3) and (6), $2x = 7 \pm 3 = 10$ or 4

Whence, $x = 5$ or 2

Subtracting (6) from (3), $2y = 7 \mp 3 = 4$ or 10

Whence, $y = 2$ or 5

EXAMPLES FOR PRACTICE.

Solve the equations,

1. $\begin{cases} x^2 + y^2 = 25 \\ xy = 12 \end{cases}$ Ans. $\begin{cases} x = \pm 4 \text{ or } \pm 3 \\ y = \pm 3 \text{ or } \pm 4 \end{cases}$

2. $\begin{cases} x - y = 6 \\ x^2 + y^2 = 90 \end{cases}$ Ans. $\begin{cases} x = 9 \text{ or } -3 \\ y = 3 \text{ or } -9 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 58 \\ x + y = -4 \end{cases}$ Ans. $\begin{cases} x = 3 \text{ or } -7 \\ y = -7 \text{ or } 3 \end{cases}$

Case III. (NOTE. This method should be employed only when the equations cannot be solved in any other way.)

ILLUSTRATIVE EXAMPLES.

Solve the equations,

$$\begin{cases} x^2 - 2xy = 5 \\ x^2 + y^2 = 29 \end{cases}$$

Putting $y = vx$ in the given equations, we have

$$x^2 - 2vx^2 = 5; \text{ or } x^2 = \frac{5}{1 - 2v} \qquad (1)$$

$$x^2 + v^2x^2 = 29; \text{ or } x^2 = \frac{29}{1 + v^2}$$

Equating the values of x^2, $\frac{5}{1-2v} = \frac{29}{1+v^2}$

$$5 + 5v^2 = 29 - 58v$$
$$5v^2 + 58v = 24$$

Solving this equation, $v = \frac{2}{5}$ or -12

Substituting in (1), $x^2 = \frac{5}{1-\frac{4}{5}}$ or $\frac{5}{1+24}$

$$x^2 = 25 \text{ or } \frac{1}{5}$$

Whence, $x = \pm 5$, or $\pm \frac{1}{\sqrt{5}}$

Substituting the values of v and x in the equation

$$y = vx$$

$$y = \frac{2}{5}(\pm 5), \text{ or } -12\left(\pm\frac{1}{\sqrt{5}}\right) = \pm 2, \text{ or } \mp\frac{12}{\sqrt{5}}$$

Ans. $x = \pm 5$, $y = \pm 2$

or, $x = \pm\frac{1}{5}\sqrt{5}$, $y = \mp\frac{12}{5}\sqrt{5}$

EXAMPLES FOR PRACTICE.

1. Solve the equations,

$$\left.\begin{aligned} x^2 - xy &= 35 \\ xy + y^2 &= 18 \end{aligned}\right\} \quad \text{Ans.} \quad \left\{\begin{aligned} x &= \pm 7, \pm\frac{5}{2}\sqrt{2} \\ y &= \pm 2, \mp\frac{9}{2}\sqrt{2} \end{aligned}\right.$$

2. Solve the equations,

$$\left.\begin{aligned} x^2 + xy &= 12 \\ xy - y^2 &= 2 \end{aligned}\right\} \quad \text{Ans.} \quad \left\{\begin{aligned} x &= \pm 3 \text{ or } \pm 2\sqrt{2} \\ y &= \pm 1 \text{ or } \pm\sqrt{2} \end{aligned}\right.$$

3. Solve the equations,

$$\left.\begin{aligned} x^2 + xy - y^2 &= 1 \\ x^2 - xy + 2y^2 &= 8 \end{aligned}\right\} \quad \text{Ans.} \quad \left\{\begin{aligned} x &= +2 \text{ or } \pm\frac{5}{11}\sqrt{11} \\ y &= \mp 1 \text{ or } \pm\frac{7}{11}\sqrt{11} \end{aligned}\right.$$

EXAMPLES FOR PRACTICE.

1. The sum of the squares of two numbers is 106, and the difference of their squares is $\frac{7}{2}$ the square of their difference. Find the numbers. Ans. 9 and 5

2. The sum of the areas of two square fields is 1,300 square rods, and it requires 200 rods of fence to enclose both. What are the areas of the fields?

Ans. 900 sq. rods; 400 sq. rods

3. The difference of the squares of two numbers is 7, and the product of their squares is 144. Find the numbers.

Ans. 3 and 4

RATIO AND PROPORTION.

45. Ratio and proportion are of great assistance in many problems.

1. **Ratio** is the relation of one quantity to another of the same kind; or it is the quotient which arises from dividing one quantity by another of the same kind. Ratio is indicated by writing the two quantities after one another with two dots between, or by expressing the division in the form of a fraction. Thus the ratio of a to b is written $a:b$, or $\frac{a}{b}$ and is read, "a is to b," or, "a divided by b."

2. The **terms** of a ratio are the quantities compared, whether single or compound. The first term of a ratio is called an *antecedent*, the other the *consequent;* and two terms together comprise a *couplet.*

3. An **inverse** or **reciprocal ratio** of any two quantities is the ratio of their *reciprocals.* Thus the *direct* ratio of a to b is $a:b$, that is, $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a}:\frac{1}{b}$, that is $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b:a$.

4. **Proportion** is an equality of ratios. Four quantities are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=), or by four dots (::). Thus $a:b = c:d$, or $a:b::c:d$, or $\frac{a}{b} = \frac{c}{d}$; read a to b equal c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

5. In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the

proportion. The first and fourth terms are called the *extremes*, and the second and third the *means*.

6. When three quantities are in proportion, e. g. $a:b = b:c$, the second is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

7. A proportion is transformed by **Alternation** when *antecedent* is compared with *antecedent*, and *consequent* with *consequent*.

8. A proportion is transformed by **Inversion** when the antecedents are made consequents and the consequents antecedents.

9. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

10. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

11. *Axiom.* Two ratios respectively equal to a third are equal to each other.

I. *In a proportion the product of the extremes is equal to the product of the means.*

Let $a:b = c:d$

that is $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions, $ad = bc$

II. *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad = bc$

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$

Whence, $a:b = c:d$

III. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second is to the fourth.*

Let $a:b = c:d$

By I, $ad = bc$

By II, $a:c = b:d$

IV. *In any proportion the terms are in proportion by* **Inversion**; *that is, the second is to the first as the fourth is to the third.*

Let $a:b = c:d$

Then, $ad = bc$

By II, $b:a = d:c$

V. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term, as the sum of the last two terms is to the third term.*

Let $a:b = c:d$

That is $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or, $$\frac{a+b}{b} = \frac{c+d}{d}$$

That is, $$a + b : b = c + d : d$$

VI. *In any proportion the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first as the difference of the last two terms is to the third.*

Let $a:b = c:d$

That is, $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each number, $\frac{a}{b} - 1 = \frac{c}{d} - 1$

or, $$\frac{a-b}{b} = \frac{c-d}{d}$$

That is, $$a - b : b = c - d : d$$

NOTE. From V and VI, and by means of III.

If $a:b = c:d$

Then $a + b : a - b = c + d : c - d$

VII. *Equimultiples of two quantities have the same ratio as the quantities themselves.*

For $\frac{a}{b} = \frac{ma}{mb}$

That is $a:b = ma:mb$

VIII. *In any proportion, like powers or like roots of the terms are in proportion.*

Hence, $\frac{a^n}{b} = \frac{c^n}{d^n}$

That is, $a^n : b^n = c^n : d^n$

NOTE. (Of course n may be either fractional or integral.)

IX. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f$

Now, $ab = ab$ (A)

And, $ad = bc$ (B)

And also, $af = be$ (C)

Adding (A), (B), (C),

$$a(b + d + f) = b(a + c + e)$$

Hence (by II), $a : b = a + c + e : b + d + f$

X. If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.

Let $a : b = c : d$

And $e : f = g : b$

By I, $ad = bc$ (A)

And $eb = fg$ (B)

Multiplying (A) by (B), $adeb = bcfg$ (C)

Dividing (A) by (B), $\frac{ad}{eb} = \frac{bc}{fz}$ (D)

From (C), $ae : bf = cg : db$

And from (D), $\frac{a}{e} : \frac{b}{f} = \frac{c}{z} : \frac{d}{b}$

NOTE. The remainder of the book is optional and may be omitted if found too difficult.

ARITHMETICAL PROGRESSION.

An arithmetical progression is a series of terms, each of which differs from the preceding by a certain definite constant quantity. Thus 2, 4, 6, 8, 10, 12 . . . . . is an increasing arithmetical progression in which the common difference is 2, for the difference of 2 and 4 is 2, the difference of 4 and 6 is 2 and so on.

Again 8, 5, 2, — 1, — 4, ... is a decreasing arithmetical progression in which the common difference is — 3, i. e., each succeeding term is found by adding — 3 to the one next preceding it.

Suppose now that we let a represent the first term, d the common difference, n the number of terms, and l the last term.

We may now represent our progression thus,

$$a,\ a + d,\ a + 2d,\ \ldots\ldots\ l - 2d,\ l - d,\ l.$$

Suppose we are given the first term a, the common difference d, and the number of terms n, and it is required to find the last term l.

It will be noted that the coefficient of d in any term is one less than the number of the term, therefore, in the last or nth term the coefficient of d will be $n - 1$. Hence,

$$l = a + (n - 1)\ d \qquad (1)$$

Given a, l and n, to find the sum of the series, S.

$$S = a + (a + d) + (a + 2d) + \ldots\ldots + (l - d) + l.$$

Writing the series in the opposite order, that is, backwards, we have,

$$S = l + (l - d) + (l - 2d) + \ldots\ldots + (a + d) + a.$$

Adding these two equations we obtain, as a result, equation (A) because the sum of corresponding terms in the series is, in each case, $a + l$.

$$2S = (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l) + (a + l) \qquad (A)$$

$$= n\ (a + l) \text{ because there are } n \text{ terms.}$$

$$\text{Hence } S = \frac{n}{2}\ (a + l) \qquad (2)$$

Substituting the value of l which we before obtained, we have

$$S = \frac{n}{2}\ [2a + (n - 1)\ d]$$

Find the last term and the sum of the series 8, 5, 2, — 1, — 4 . . . . to 27 terms.

In this series $a = 8$, $d = -3$, and $n = 27$.

Substituting in (1) and (2) we have

$$l = 8 + (27 - 1)\ (-3) = 8 - 78 = -70$$

$$S = \frac{27}{2}(8 - 70) = 27 \times (-31) = -837$$

EXAMPLE FOR PRACTICE.

Find last term and sum in following series:

1. 7, 3, — 1 . . . . to 20 terms. Ans. $\begin{cases} l = -69 \\ S = -620 \end{cases}$

2. — 5, — 10, — 15 . . . . to 29 terms.

Ans. $\begin{cases} l = -145 \\ S = -2175 \end{cases}$

3. $\frac{3}{5}, \frac{8}{15}$ . . . . to 19 terms. Ans. $\begin{cases} l = -\frac{3}{5} \\ S = 0 \end{cases}$

If any three of the five elements of an arithmetical progression be given, the other two may be found by substituting the known values in the fundamental formulæ (1) and (2), and solving the resulting equations.

For example:

Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (1) and (2)

$$-39 = a + (n-1)(-3) \text{ or } a = 3n - 42$$

Substituting this value of a in the following:

$$-264 = \frac{n}{2}(a - 39), \text{ or } an - 39n = -528$$

We have $3n^2 - 42n - 39n = -528$

Or $n^2 - 27n = -176$

Solving $n = 16$ or 11

Hence as $a = 3n - 42$, we have $a = 48 - 42$ or $33 - 42 = 6$ or -9.

Ans. Two solutions $a = 6$ and $n = 16$, or $a = -9$, and $n = 11$

Accordingly the series would be

If $a = 6$ and $n = 16$

6, 3, 0, — 3, — 6 . . . . — 33, — 36, — 39

If $a = -9$ and $n = 11$

— 9, — 12, — 15, — 18, — 21 . . . . — 33, — 36 — 39

As is seen the last term in each of these series is — 39 and the sum — 264.

Insert 5 arithmetical means between 3 and — 5.

This signifies that we are to find 7 terms in arithmetical progression, such that the first term is 3 and the last — 5.

Therefore $a = 3$, $l = -5$, and $n = 7$

Substituting in (1),

$$-5 = 3 + 6d$$

Hence $d = -\frac{4}{3}$ and the required series is

$$3, \frac{5}{3}, \frac{1}{3}, -1, -\frac{7}{3}, -\frac{11}{3}, -5$$

EXAMPLES FOR PRACTICE.

Insert 4 arithmetical means between -1 and -7.

Ans. $d = -\frac{6}{5}$ and series is $-1, -\frac{11}{5}, -\frac{17}{5}, -\frac{23}{5}$. $-\frac{29}{5}, -7$.

Find four quantities in arithmetical progression such that the product of the extremes shall be 45, and the product of the means 77.

Let the quantities be $x - 3y$, $x - y$, $x + y$, and $x + 3y$. Then, by the conditions,

$$\begin{cases} x^2 - 9y^2 = 45 \\ x^2 - y^2 = 77 \end{cases}$$

Solving these equations, $x = \pm 9$, and $y = \pm 2$. Therefore the quantities are 3, 7, 11, and 15; or -3, -7, -11, and -15.

NOTE. The student will find it convenient in such a problem as the above, to let *symmetrical* expressions represent the unknown quantities. For instance, to represent *five* quantities, we should have used $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

EXAMPLES FOR PRACTICE.

1. Find the sum of the even numbers beginning with 2 and ending with 500. Ans. 62,750

2. A falling body descends at the rate of $16\frac{1}{12}$ feet for the first second, and thereafter gains in velocity $32\frac{1}{6}$ feet in each succeeding second. How far will it fall in 16 seconds?

Ans. $4,117\frac{1}{3}$ feet.

3. Two travellers start together. A travels at the uniform rate of 10 miles a day, whereas B covers but 8 miles the first day, increasing his speed however by half a mile upon each succeeding day. How far from their starting point, and after how many days will they come together?

Ans. At a distance of 90 miles and after nine days.

GEOMETRICAL PROGRESSION.

A geometrical progression is a series of terms in which each succeeding term is derived from the one immediately preceding it, by multiplying that preceding term by a constant quantity called the *ratio*.

Thus 2, 4, 8, 16, 32 . . . . . is an increasing geometrical progression in which the ratio is 2, i.e., each term is twice the term immediately preceding it.

Again 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$ . . . . . is a decreasing geometrical progression in which the ratio is $\frac{1}{3}$, each term being $\frac{1}{3}$ of the preceding term.

We may also have a negative value for the ratio; thus — 3, 9, — 27, 81, — 243 is a progression in which the ratio is — 3, thus causing the positive and negative signs to alternate.

Suppose, now, that we are given the first term, a, the ratio, r, and the number of terms, n, to find the last term l.

We may write the progression thus,

$$a,\ ar,\ ar^2,\ ar^3, \ldots\ldots\ldots$$

By observing this we see that the exponent of r is, in any term, one less than the number of the term. Hence in the last or nth term, the exponent of r will be $n - 1$. That is,

$$l = ar^{n-1} \qquad (1)$$

Given the first term, a, the last term, l, and the ratio, r, to find the sum of the series S.

$$S = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1}$$

Multiplying each term by r, we have

$$rS = ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$$

Subtracting the first of these equations from the second, we have, as all but two terms cancel, the first, second, third, etc., of the second cancelling the second, third, fourth, etc., of the first,

$$rS - S = ar^n - a$$

or,

$$S(r-1) = ar^n - a$$

therefore,

$$S = \frac{ar^n - a}{r-1}$$

and, as $l = ar^{n-1}$, $rl = ar^n$; that is $ar^n = rl$ and substituting rl for ar^n we have

$$S = \frac{rl - a}{r-1} \qquad (2)$$

ILLUSTRATIVE EXAMPLE.

In the series $-2, 6, -18, 54, \ldots$ to 8 terms, find the last term and the sum.

In this case $a = -2$, $r = -3$, $n = 8$. Hence,

$$l = -2(-3)^7 = -2 \times (-2187) = 4374$$

$$S = \frac{-3 \times 4374 - (-2)}{-3-1} = \frac{-13122+2}{-4} = 3280$$

In each of the following find the last term and the sum of the series.

1. $1, 2, 4, \ldots$ to 9 terms. Ans. $l = 256$, $S = 511$

2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, to 11 terms. Ans. $l = \frac{1}{2048}$, $S = \frac{2047}{2048}$

3. $3, -6, 12, \ldots$ to 7 terms. Ans. $l = 192$, $S = 129$

If any three of the five elements of a geometrical equation are given, the other elements may be found by substituting the known values and solving the equation. The equation thus formed may be of higher degree than the second, and others require the use of logarithms which are treated later on in the course. Examples of this nature are, therefore, omitted here.

The sum of the terms of a *decreasing* geometrical progression approaches a limit as the number of terms is indefinitely increased. This limit is called the *sum of the series to infinity*.

The value of S in the formula previously given is $\frac{rl - a}{r - 1}$

This may be written thus:

$$S = \frac{a - rl}{1 - r}$$

In a *decreasing* geometrical progression the greater the number of terms taken, the smaller the value of the last term. As the number of terms is increased indefinitely, the quantity rl is decreased indefinitely, approaching 0 as a limit.

Hence the fraction $\frac{a - rl}{1 - r}$ approaches $\frac{a - 0}{1 - r}$ or $\frac{a}{1 - r}$

Hence, the sum of a decreasing geometrical progression is given by the following formula:

$$S = \frac{a}{1 - r}$$

Find the sum of the series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, ... to infinity.

In this series $a = 2$ and $r = \frac{1}{2}$

Hence,

$$S = \frac{2}{1 - \frac{1}{2}}$$

$$= \frac{2}{\frac{1}{2}}$$

$$= 4$$

Hence the sum to infinity is 4.

TO DETERMINE THE VALUE OF A REPEATING DECIMAL.

In this case we use the formula for the sum to infinity of a geometrical progression ($S = \frac{a}{1 - r}$)

Find the value of .52121 . . . .

$$.52121 \ldots . = .5 + .021 + .00021 \ldots .$$

The terms after the first give a geometrical progression in which $a = .021$ and $r = .01$.

Hence by the formula the sum of the progression is

$$S = \frac{.021}{1 - .01} = \frac{.021}{.99} = \frac{21}{990} = \frac{7}{330}$$

The value of .5 is $\frac{1}{2}$

Hence the value of .52121 . . . . is

$$\frac{1}{2} + \frac{7}{330} = \frac{172}{330}$$
$$= \frac{86}{165}$$

EXAMPLES FOR PRACTICE.

Find the values of:

1. .407407 Ans. $\frac{11}{27}$

2. .7333 Ans. $\frac{11}{15}$

Insert 4 geometrical means between 2 and $\frac{64}{243}$

This signifies that we are to find 6 terms in geometrical progression such that the first term is 2 and the last $\frac{64}{243}$.

Substituting in (1) the values $a = 2, n = 6, l = \frac{64}{243}$ we have

$$\frac{64}{243} = 2r^5 \text{ whence } r^5 = \frac{32}{243} \text{ or } r = \frac{2}{3}$$

Hence the required series is $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}$

EXAMPLE FOR PRACTICE.

Insert 6 geometrical means between — 2 and — 4374.

Ans. $r = 3$ and series is — 2, — 6, — 18, — 54, — 162, — 486, — 1458, — 4374.

BINOMIAL THEOREM.

The **binomial theorem** is a formula by means of which any power of a binomial may be expanded into a series. In the following only those cases in which the exponent is a positive integer are considered.

Expanding the following by actual multiplication, we have,

$(a + x)^2 = a^2 + 2ax + x^2$

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$

$(a + x)^5 = a^5 + 5a^4x + 10a^3x^3 + 10\,a^2x^3 + 5ax^4 + x^5$

and so on.

By a consideration of the above we obtain the following laws:

I. The number of terms is greator by 1 than the exponent of the binomial.

II. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

III. The exponent of x in the second term is 1 and increases by 1 in each succeeding term.

IV. The coefficient of a in the first term is 1, the coefficient of the second term is the power to which the binomial is raised. In the third it is the power to which the binomial is raised times that quantity minus 1 divided by 1 times 2, and so on as will become apparent from the following:

Let n represent the power to which the binomial $a + x$ is to be raised.

Then

$$(a + x)^n = a^n + na^{n-1}x + \frac{n\,(n-1)}{1\,.\,2}a^{n-2}x^2 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}x^4 \ldots .$$

and so on to

$$\ldots . \frac{n\,(n-1)\,(n-2)\ldots .3\,.\,2\,.\,1}{1\,.\,2\,.\,3\ldots .(n-2)\,(n-1)\,n}x^n$$

In the last term all the factors in both numerator and denominator cancel giving x^n (the exponent of a being $n - n$ or 0 and $a^0 = 1$; hence a disappears from the last term of the series.) Thus for our last term we have x^n just as we had a^n in the first term. To illustrate: In $(a + x)^4$ the first term was a^4, the last x^4. In $(a + x)^5$ the first term was a^5, the last x^5. In $(a + x)^6$

expanded the coefficient of the second term is 5, that of the third $\frac{5 \cdot 4}{1 \cdot 2}$ or 10, that of the fourth $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$ or 10, that of the fifth, $\frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}$ or 5, that of the sixth and last $\frac{5 \cdot 4 \cdot 2 \cdot 3 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 1, all the factors cancelling. It is to be further noted that the *sum* of the exponents of the first and second factors in any term of an expansion of a binomial (as a and x) is equal to the power to which the binomial is to be raised. If x is negative the terms of the expansion are alternately positive and negative. Thus,

$$(a - x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 \ldots .$$

If $a = 1$, we have for $(a + x)^n$,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3$$

Suppose we wish to expand $(x - y)^7$.

$$\begin{aligned}(x - y)^7 &= x^7 - 7x^6y + \frac{7 \cdot 6}{1 \cdot 2} x^5y^2 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} x^4y^3 \\ &\quad + \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} x^3y^4 - \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^2y^5 \\ &\quad + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} xy^6 - \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} y^7 \\ &= x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 \\ &\quad + 7xy^6 - y^7\end{aligned}$$

EXAMPLES FOR PRACTICE.

Expand the following:

1. $(m - n)^5$

   Ans. $m^5 - 5m^4n + 10m^3n^2 - 10m^2n^3 + 5mn^4 - n^5$

2. $\left(\frac{x}{y} - \frac{3y}{x}\right)^3$ Ans. $\frac{x^3}{y^3} - \frac{9x}{y} + \frac{27y}{x} - \frac{27y^3}{x^3}$

   or $x^3y^{-3} - 9xy^{-1} + 27x^{-1}y - 27x^{-3}y^3$

A trinomial may be raised to any power by the binomial theorem if two of its terms be enclosed in a parenthesis and treated as a single term.

TO FIND ANY TERM IN AN EXPANSION.

The following laws will be found to hold good for any term in the expansion of $(a + x)^n$.

I. The exponent of x is the number of the term decreased by 1.

II. The exponent of a is n minus the exponent of x.

III. The last factor of the numerator of the coefficient is greater by 1 than the exponent of a.

IV. The last factor of the denominator is the same as the exponent of x.

Hence in the 12th term,

The exponent of x will be $12 - 1$ or 11.

The exponent of a will be $n - (12 - 1)$ or $n - 11$.

The last factor of the numerator will be $n - 10$.

The last factor of the denominator will be 11.

If r represents the required term then in the rth term,

The exponent of x will be $r - 1$.

The exponent of a will be $n - (r - 1)$ or $n - r + 1$.

The last factor of the numerator will be $n - r + 1$.

The last factor of the denominator will be $r - 2$.

Thus the rth term is

$$\frac{n\,(n-1)\,(n-2)\,\ldots\,(n-r+2)}{1\,.\,2\,.\,3\,\ldots\,(r-1)}\,a^{n-r+1}\,x^{r-1}$$

Let us find the 8th term of $(3x^2 - y^{-1})^{11}$

$$= \frac{11\,.\,10\,.\,9\,.\,8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}\,(3x^2)^4\,(-y^{-1})^7$$

$$= 330\,(81x^8)\,(-y^{-7}) = -26730x^8y^{-7} \qquad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the

1. Eighth term of $(a - b)^{12}$ Ans. $-792a^5b^7$
2. Fifth term of $(1 - x^2)^{14}$ Ans. $1001x^8$

NOTE. The odd terms of an expansion are positive, the even terms negative.

APPENDIX

QUESTIONS IN REVIEW

The following lists of questions afford the reader a means of testing his knowledge of the subjects treated in the present volume. Any one who has carefully read the foregoing subjects should be able to answer every question accurately. The reader is advised to go through the questions thoroughly, working out his answers and checking his results with the answers given. This will afford a valuable means of fixing the matter in mind. Students who are preparing for examinations for college, civil service or engineer's license will find the questions of the greatest value.

REVIEW QUESTIONS

ON THE SUBJECT OF

ARITHMETIC.

EXERCISE I.

1. What is Arithmetic?
2. What is a unit of a number?
3. When is a number concrete?
4. What is numeration?
5. Write in words the following numbers: 25; 340; 1,709; 28,641,811.
6. Write in figures, one hundred eighty; one thousand six hundred forty-seven; eight million nine hundred nine thousand two hundred six.

EXERCISE II. (Cancellation.)

1. A laborer gave 12 days' work for 48 bushels of potatoes, worth 50 cents a bushel. What were his daily earnings?

Ans. $2.00.

2. Sold 20 pounds of butter at 27 cents a pound, which exactly paid for 15 pounds of coffee. What was the price of the coffee a pound? Ans. $.36.

3. How many bushels of oats at $.60 a bushel, will pay for 12 tons of coal at $7.20 a ton? Ans. 144 bushels.

4. A grocer exchanged 462 pounds of sugar, at 14 cents per pound, for eggs at 22 cents per dozen. How many dozen were received? Ans. 294 dozen.

5. A ship has provision for 30 men 12 months. How long will it last 45 men? Ans. 8 months.

MISCELLANEOUS PROBLEMS.

1. The armature of a dynamo revolves 780 times per minute. How many times will it revolve in 6 hours (360 minutes)?

Ans. 280,800 times.

2. 33,000 foot pounds per minute is one horse-power. How many foot pounds per minute are there in 17 horse-power?

Ans. 561,000 foot-pounds.

3. An electric light plant for 200 sixteen-candle-power lamps cost as follows:

16 horse-power engine,	1,770 dollars,
Dynamo,	625 "
Fixing dynamo and engine,	35 "
Driving-belt for Dynamo,	50 "

What is the total cost? Ans. 2,480 dollars.

4. If one Grove cell furnishes 2 volts, how many volts will 17 cells furnish? Ans. 34 volts.

5. There are 360 rivets in a hundred pounds. How many pounds will be required to afford 64,800 rivets?

Ans. 18,000 pounds.

6. A locomotive has 4 driving wheels. The total weight is 133,000 pounds. What should be the weight on each?

Ans. 33,250 pounds.

7. A nine-foot blower makes 175 revolutions per minute. How many revolutions will it make in 29 minutes?

Ans. 5,075 revolutions.

8. 20 cubic feet of wood weigh 1,260 pounds. How much do 10 cubic feet weigh? Ans. 630 pounds.

9. Supposing a cubic foot of anthracite coal to weigh 55 pounds. How many cubic feet will there be in 1100 pounds?

Ans. 20 cubic feet.

10. 5 boilers have a heating surface of 15,770 square feet. How many square feet has each? Ans. 3,154 square feet.

11, California contains 158,933 square miles, and Texas 237,321 square miles. How much larger is Texas than California? Ans. 78,388 square miles.

12. What will be the cost of building a line of telegraph 274 miles long, at 967 dollars per mile? Ans. 264,958 dollars.

13. If it requires 108 tons of iron rail for one mile of track, how many tons will be required for 476 miles? Ans. 51,408 tons.

14. The sailing distance from New York to Queenstown is 2,890 miles. If a steamer has run 1,279 miles on the course, how far has she to run? Ans. 1,611 miles.

15. From 946 subtract 486; from the result subtract 284; to this add 398; from this result subtract 199. Ans. 375.

16. A Guibal double mine ventilating fan runs at the rate of 84 revolutions per minute. It gives 2,818 cubic feet of air per revolution. How many cubic feet should it give per minute?
Ans. 236,712 cu. ft.

17. A roof is composed of eleven frames. The weight of each in detail is as follows:

2 rafters,	each weighing	875 pounds.
5 rods,	"	176 "
16 bolts,	"	5 "
8 bridle straps,	"	15 "
2 piers connecting rafter at ridge,	average each,	11 "
6 pieces at foot of strut,	" "	11 "
4 pieces uniting rafters at junction in strut, with bolts and nuts,	each	44 "
2 cast iron struts,	"	154 "
2 rafter shoes,	"	144 "
Total weight?		Ans. 40,590 pounds.

18. A tank contains 1,348,192,450 cubic feet of water. The loss by evaporation, etc., is 233,220,240 cubic feet. How much remains? Ans. 1,114,972,210 cubic feet.

19. If a gas engine uses 20 cubic feet of gas per horse-power per hour, how many cubic feet of gas will be required to run a 12 horse-power gas engine an hour? Ans. 240 cubic feet.

20. A contractor engages to build a wall 1,540 feet long in 19 days, and hires 30 men. At the end of 11 days he finds only 660 feet of the wall completed. How many additional men must he employ to finish the work in the stipulated time?

Ans. 25 additional men.

21. A certain boiler test showed the following amounts of coal used:

Monday	720 pounds
Tuesday	712 "
Wednesday	718 "
Thusrday	722 "
Friday	714 "
Saturday	704 "

What was the average coal consumption per day?

Ans. 715 pounds.

22. An Amoskeag steam fire-engine weighs 7,522 pounds. After water is put in the boiler the weight becomes 8,914 pounds. If water weighs 8 pounds per gallon, how many gallons of water were in the boiler? Ans. 174 gallons.

23. A stone crusher is capable of yielding 27 cubic feet of broken stone per hour. If a cubic foot of stone weighs, on the average, 97 pounds, what weight of stone will be yielded in 4 hours? Ans. 10,476 pounds of stone.

24. A Corliss engine at the Centennial exhibition made 2,355,300 revolutions. If it made 36 revolutions per minute, how many hours (60 minutes) with how many minutes over an even number of hours did it run? Ans. 1090 hours 25 minutes.

25. To produce a long ton (2,240 pounds) of pig iron the following weights of fuel were consumed:

Charcoal,	2,209 pounds
Anthracite,	2,964 "
Coke,	2,421 "
Raw coal,	3,606 "

On this basis what amount of fuel is required to produce a pound of pig iron? Ans. 5 lbs.

26. A 12-inch gas main delivers 65,936 cubic feet per hour. How many feet will it deliver in 14 hours?

Ans. 923,104 cubic feet.

27. A pump delivers 33,720 cubic feet of water in two hours (60 minutes each.) How many cubic feet are delivered per minute? Ans. 281 cubic feet.

28. The weight in detail of a steam hammer is as follows:

Tup with piston and rod,	40,000	pounds
Cylinder,	11,000	"
Entablature,	15,000	"
Leg of hammer with guides,	125,000	"
Wrought iron cross-stays,	12,000	"
Foundation-plate,	45,000	"
Valves, valve-gear, etc.,	16,500	"
Anvil and anvil-block,	37,500	"

What is the total weight in tons (2,000 pounds each)?

Ans. 151 tons.

29. Two internally fired tubular boilers with return flues beneath the shells have each the following heating surfaces:

Heating surface in external shell,	173	square feet
" " " furnaces,	115	" "
" " " combustion chamber,	158	" "
" " " each of 80 tubes,	7	" "

What is the total heating surface of both boilers?

Ans. 2,012 square feet.

30. Single belting 36 inches wide costs 7 dollars per running foot. Double belting costs twice as much. What is the cost of 198 feet of double belting? Ans. $2,772 cost.

31. What is the total amount of water per hour used by an engine of 213 effective horse-power if it uses 19 pounds per effective horse-power per hour? Ans. 4,047 pounds.

32. If 22 pounds of water per hour per electrical horse-power are used, how many pounds of water per hour are required to furnish 213 horse-power? On the basis of this and the preceding example, how many more pounds are required to furnish 213 electrical horse-power than 213 direct from engine?

Ans. { 4,686 pounds in first case.
639 pounds in second case.

33. The following bids for paving (with brick) Kansas Ave. (Atchison) from Fourth to Tenth Streets, were received:

Atchison Paving Brick Co.,	$12,226
J. M. Lane & Co.,	12,508
O. W. Ulrich,	12,619
J. M. Iddings,	12,627
A. L. Kimball,	12,756
J. G. Thayer,	13,518

What was the average bid? Ans. $12,709.

(Suggestion. Add and divide by 6.)

34. The heating surface in a locomotive is 88 square feet in the fire box and 792 square feet in the tubes. What is the total heating surface? How many times as much surface is there in the tubes as there is in the fire box?

Ans. { 880 square ft. / 9 times.

35. The weight of a car and passengers is as follows:

Weight of car body,	20,000	pounds
" " both trucks,	20,000	"
" " two motors,	7,000	"
" " 75 passengers,	9,000	"

What is the total weight in tons (2,000 pounds)?

Ans. 28 tons.

36. A steam plant has two engines of 934 horse-power each. The starboard engine of a steam ship develops 3218 horse-power and the port engine 3232 horse-power. How much greater power is developed in the ship than in the steam plant.

Ans. 4582 horse-power.

37. The weight of a battery of eight marine boilers and their equipments is as follows:

Boilers complete with mountings,	295 tons.
Water in boilers,	73 "
Funnels,	50 "
Stoke-hole plates, floors, etc.,	23 "
Feed pumps,	7 "
Fans and fan engines,	8 "
Feed regulators,	2 "
Tools and fittings,	2 "
Spare gear,	10 "

What is the weight of two such batteries? Ans. 940 tons.

38. Each of the above boilers has a grate surface of 95 square feet. What is the total grate surface of the two batteries?

Ans. 1,520 square feet.

39. A ferry-boat uses 1,560 pounds of coal per hour. The amount of ashes taken from the ash-pit is 171 pounds per hour. What weight of coal goes off through the funnel as gas, etc., in 7 hours? Ans. 9,723 pounds.

40. A dynamo requires 30 horse-power to drive it. It furnishes 18,650 watts. There are 746 watts in a horse-power. How many horse-power does it furnish? Ans. 25 horse-power.

41. If it costs an electric road three cents per passenger for transportation, how many dollars does the road make by carrying 487,600 passengers at 5 cents each (supposing that all of the passengers pay)? Ans. $9,752.

42. The amount of water flowing through a 6-inch turbine is 59 cubic feet per minute. How much water will flow through it in six days (each 24 hours of 60 minutes)?

Ans. 509,760 cubic feet.

43. Assuming the pressure of the atmosphere to be 15 pounds per square inch, what is the atmospheric pressure against a piston 113 square inches in area? What is the total pressure on a square foot, or 144 square inches?

Ans. { 1695 pounds. / 2160 pounds.

44. A ship displaces 8,270 tons. Supposing 35 cubic feet of sea water to weigh one ton, how many cubic feet would she displace? Ans. 289,450 cubic feet.

45. At 100 degrees Fahrenheit a cubic foot of water weighs 62 pounds. At 205 degrees a cubic foot weighs 60 pounds. What is the difference in weight between 173 cubic feet of water at 100 degrees and 189 cubic feet at 205 degrees? Ans. 614 pounds.

46. A centrifugal pump delivers 2,371 gallons per minute. A gallon weighs 8 pounds. What weight of water is delivered per minute? Ans. 18,968 pounds.

47. A ship has 360 tons of coal in her bunkers. After steaming 1,360 miles she has consumed 170 tons. How many tons has she after steaming 2,320 miles? Ans. 70 tons.

48. There are 357 pages in a book. The book is divided into 17 chapters. How many pages are there in a chapter, on the average? Ans. 21 pages.

49. How many kegs of nails, each weighing 100 pounds, are there in three tons (2,000 pounds each)? Ans. 60 kegs.

50. A steamer runs 3,838 miles in 8 days (24 hours each), 10 hours. What is her average speed per hour? How long would it take her to steam 4,788 miles at the same rate?

Ans. { 19 miles per hour.
252 hours.

51. How many minutes are there in 4 days (24 hours each, and 60 minutes in an hour)? How many seconds (60 in each minute) in the same amount of time?

Ans. { 5,760 minutes.
345,600 seconds.

52. A fly-wheel has a circumference of 30 feet. How far does a point on the edge of it travel in 941 revolutions?

Ans. 28,230 feet.

53. Add 1,743 and 11,469, multiply by 13, divide by 3 and subtract 12,608. Ans. 44,644.

54. A town has 2,549 voters. There are twice as many inhabitants who do not vote. What is the population?

Ans. 7,647 people.

55. How great is the difference between 1,842 divided by 3 and 11,837 divided by 7? Ans. 1,077.

56. How many times is 1,411 contained in 11,288? Ans. 8.

57. Multiply 706 by itself and multiply the result by 706. (This is called the cube of 706.) Ans. 351,895,816.

58. Divide 9,807 by 3. Multiply the quotient by 7, subtract 8,493 and add 2,014. Ans. 16,404.

59. A team is driven 45 miles in 9 hours. A mile is 5,280 feet. If the wheels measure 4 feet on the circumference, *i. e.*, around the tire, how many revolutions do they make in an hour?

Ans. 6,600 revolutions.

REVIEW QUESTIONS

ON THE SUBJECT OF

ARITHMETIC.

EXERCISE I.

1. What is a fraction?
2. What does the line between the terms of a fraction mean?
3. What does the denominator of a fraction show?
4. What does the numerator show?
5. What is the value of a fraction?

EXERCISE II.

1. What is a proper fraction? An improper fraction? Give two examples of proper fractions. Give two of improper fractions.
2. What is a mixed number? Give an example.
3. How is a fraction reduced to higher terms? To lower terms? Give an example of each.
4. If the numerator is increased, what effect will it have on the value of a fraction? If the denominator is increased?
5. What is the effect on the value of a fraction if the numerator is decreased? If the denominator is decreased?

EXERCISE III.

1. What effect would it have on the value of a fraction to divide the numerator? To divide the denominator?
2. What effect would it have on the value of a fraction to multiply the numerator? To multiply the denominator?
3. If we multiply both numerator and denominator of a fraction by 17 and then divide each product by 13, would it produce a change in the value of the fraction?

EXERCISE IV.

1. Change $\frac{7}{12}$ to a fraction whose denominator is 96.
2. Change $\frac{11}{15}$ to a fraction whose denominator is 105.
3. Reduce $\frac{72}{100}$ to lowest terms.
4. Reduce $\frac{75}{125}$ to lowest terms.
5. Reduce $\frac{1344}{1536}$ to lowest terms.
6. Change $184\frac{32}{95}$ to an improper fraction.
7. Change $\frac{218}{9}$ to a mixed number.
8. Reduce $\frac{344}{536}$ to lowest terms.
9. Change $\frac{2431}{85}$ to a mixed number.
10. Change $\frac{2828}{46}\frac{80}{4}$ to a mixed number.

EXERCISE V.

Find the sum of:

1. $\frac{4}{7} + \frac{9}{10} + \frac{1}{2} + \frac{3}{5}$
2. $\frac{5}{16} + \frac{3}{30} + \frac{1}{4} + \frac{3}{5}$
3. $\frac{2}{25} + \frac{4}{21} + \frac{7}{15} + \frac{12}{35}$
4. $1\frac{4}{13} + 7\frac{9}{11}$
5. Five lengths of shafting weigh respectively $339\frac{4}{5}$, $409\frac{1}{2}$, $387\frac{1}{4}$, $418\frac{3}{101}$, and $357\frac{3}{20}$ pounds. What is their total weight?

EXERCISE VI.

Find the values of the following:

1. $\frac{5}{12} - \frac{2}{21}$
2. $\frac{11}{30} - \frac{7}{27}$
3. $\frac{7}{18} - \frac{4}{39}$
4. $19 - 2\frac{1}{8}$
5. $17\frac{1}{2} - 12\frac{9}{10}$

EXERCISE VII.

1. Two boats are a mile (5280 feet) apart. They row towards each other, one at the rate of $323\frac{7}{20}$ feet per minute, the other at the rate of $311\frac{17}{70}$ feet per minute. How far apart are they at the end of two minutes? Ans. $4010\frac{57}{70}$ feet.

2. A dynamo requires $37\frac{3}{4}$ horse-power to drive it. If it loses $1\frac{7}{8}$ horse-power what horse-power does it furnish?

Ans. $35\frac{7}{8}$ horse-power.

3. Supposing a Grove cell to have an electromotive force of $1\frac{9}{10}$, how much greater is its electromotive force than that of a Daniell cell if the latter has $1\frac{3}{45}$? Ans. $\frac{5}{6}$.

EXERCISE VIII.

Multiply

1. $\frac{9}{10}$ and $\frac{5}{21}$
2. $\frac{8}{15}$ and $\frac{9}{20}$
3. $376\frac{22}{68}$ and 21
4. $8\frac{4}{10}$ and $24\frac{1}{7}$
5. $\frac{4}{13}$ and $\frac{1}{12}$ and $\frac{3}{4}$ and $6\frac{4}{6}$

Divide

6. $\frac{120}{121}$ by 72
7. $5\frac{1}{4\frac{1}{2}}$ by 11
8. $\frac{5}{7}$ by $\frac{15}{28}$
9. $\frac{12}{45}$ by $\frac{14}{15}$
10. 1 by $541\frac{3}{4}$

EXERCISE IX.

1. What is the cost of $22\frac{3}{4}$ thousand bricks at $\$11\frac{3}{4}$ per thousand? Ans. $\$267.31 +$

2. What distance can a vessel sail in 37 hours, going at the rate of $5\frac{5}{6}$ miles per hour? Ans. $215\frac{5}{6}$ miles.

3. If 11 men can do a piece of work in $47\frac{3}{8}$ days, how long would it take 3 men to do it if each works at the same rate?

Ans. $174\frac{7}{8}$ days.

4. A stone crusher yields on the average $23\frac{7}{8}$ cubic feet of broken stone per hour. If the average weight of the broken stone per cubic foot is $93\frac{7}{12}$ pounds, what weight of stone will the crusher yield in $17\frac{3}{4}$ hours? Ans. $39658\frac{33}{384}$ pounds.

EXERCISE X.

1. How is the denominator of a decimal fraction expressed?

2. How is a fraction changed to a decimal? Give an example.

Write the following fractions as decimals:

3. $\frac{9}{11}$
4. $\frac{3}{4}$
5. $\frac{1}{15}$
6. $\frac{13}{17}$
7. $\frac{411}{700}$

NOTE. Carry out to four decimal places.

EXERCISE XI.

Find the sum of

1. .3 + .6 + .42 + .39 + .27
2. .648 + .016 + .621 + .108 + .1881
3. .428571 + 5.71428 + 1.86
4. 139.273 + .0037 + 5.91144
5. 9,843,218.431 + 716.000814 + .335

Subtract

6. .847 from .923
7. 438.58719 from 1,229.68227
8. 114.00076 from 203
9. A square foot of $\frac{5}{8}''$ wrought iron plate weighs 25.176 pounds. The same size plate in cast iron weighs 23.466. What is the difference in weight per square foot of the two plates?

Ans. 1.710 pounds.

10. A single-acting pump with 12-inch cylinder and 14-inch stroke delivers 68.544 gallons per minute. A similar pump with 16-inch stroke delivers 78.336 gallons per minute. How much more per minute does the latter deliver? Ans. 9.792 gallons.

EXERCISE XII.

Multiply

1. .075 by 13
2. .947 by .0016
3. 4001.876 by 1.00003
4. A square foot of wrought iron plate, $\frac{9}{16}$ of an inch in thickness, weighs 22.659 pounds. How much will $17\frac{3}{4}$ square feet weigh? Ans. 402.19725 pounds.
5. A three-inch pipe weighs 7.536 pounds per linear foot. What will 740.75 feet weigh? Ans. 5582.292 pounds.

EXERCISE XIII.

Divide

1. .00468 by .013
2. 43.287 by 10.103
3. .48179 by 37.2
4. The weight of 750 rivets (2 by $\frac{5}{8}$ inches), is 191.1 pounds. What is the weight of a single rivet?

Ans. .2548 pound.

MISCELLANEOUS PROBLEMS.

1. The revolutions per minute of an engine were as follows: 40, 45, 40, 46, 38.67, 38.5, 38.6. What was the average number of revolutions per minute? Ans. 40.9671 +

2. An investigation of the value of tan as fuel showed that a pound of wet tan evaporated .893 pound of water (from 212° F.) A pound of dry tan was found to evaporate 2.319 pounds. How much more water did the tan evaporate when dry?
Ans. 1.426 pounds.

3. In the case of an air compressor, the amount of air in cubic feet passing through the valves per hour is as follows: 111.864, 125.844, 118.632, 132.534, 116.434, 115.818, and 117.74. What is the average number of cubic feet? Ans. 119.838 cubic feet.

4. An engine makes 113.9 revolutions per minute. How many revolutions will it make in 2.5 hours (60 minutes each)?
Ans. 17085 revolutions.

5. The specific gravity of cast copper is 8.94. That of cast iron is 7.207. How much greater is the specific gravity of cast copper than that of cast iron? Ans. 1.733

6. A locomotive has a total coal capacity of 8,000 pounds. After using $2\frac{7}{8}$ tons (2000 pounds each) how many pounds remain? Ans. 2250 pounds.

7. One kind of lubricating oil costs $ .90 per gallon. Another kind costs $ .43. What is the total difference in price of $7\frac{3}{8}$ gallons of each? Ans. $3.46\frac{5}{8}$.

8. A steam pump delivers 26.44 gallons per stroke. A gallon weighs 8.355 pounds. What weight of water will it deliver in 117 strokes? Ans. 25846.0254 lbs.

9. The external diameter of a tube is 12 inches. The iron is .24 of an inch thick, what is the internal diameter?
Ans. 11.52 inches.

10. It is 11:07 A. M. in Chicago when it is 12:12 P. M. in Boston. What is the difference in time between the two places?
Ans. 1 hour 5 minutes.

11. The number of watts is equal to the number of amperes multiplied by the number of volts. There are 746 watts in a horse-power. How many amperes are required to furnish $16\frac{1}{2}$ horse-power, the number of volts being 500?
Ans. 24.618 amperes.

REVIEW QUESTIONS

ON THE SUBJECT OF

ARITHMETIC.

EXERCISE I.

1. What is 11% of 17,900? Ans. 1,969.
2. What is $\frac{2}{5}$% of 485? Ans. 1.94.
3. What is $9\frac{3}{4}$% of 161,218? Ans. 15,718.75$\frac{1}{2}$.
4. What is $113\frac{1}{2}$% of 149,864? Ans. 170,095.64.

EXERCISE II.

1. What per cent of $17,643 is $91? Ans. .5+%.
2. What per cent of 9,711,461 is 7? Ans. .00007+%.
3. What per cent of $\frac{11}{16}$ is $\frac{43}{681}$? Ans. 9+%.

EXERCISE III.

1. The base is $177.50 and the percentage $11.04. What is the rate? Ans. $6\frac{78}{355}$%.
2. The percentage is 17.47 and the rate 7%. What is the base? Ans. 249.57$\frac{1}{7}$.
3. A lot of coal was bought for $791. For what must it be sold in order to make 31%? Ans. $1,036.21.

EXERCISE IV.

Reduce the following:

1. 7 T. 376 lbs. 7 oz. to ounces. Ans. 230,023 ounces.
2. 97° 37′ 49″ to seconds. Ans. 351,469 seconds.
3. If a wheel has a circumference of 6.75 feet how many times will it revolve in going $7\frac{1}{4}$ miles? Ans. 5,671$\frac{1}{3}$ times.
4. What is the capacity in gallons of a tank that will hold 3 hhds. 1 bbl. 13 gals. 3 qts.? Ans. 234$\frac{1}{4}$ gallons.

EXERCISE V.

Change to higher denominations:

1. 34,741 gills. Ans. 17 hhds. 14 gals. 2 qts. 1 pt. 1 gi.
2. 187,439 pints. Ans. 2928 bus. 2 pks. 7 qts. 1 pt.
3. 84,793,112 ounces. Ans. 2649 T. 15 cwt. 69 lbs. 8 oz.
4. What is the weight of 352 gills of water?

Ans. 91.63 lbs.

EXERCISE VI.

What is the sum of the following?

1. 8° 17′ 32″ and 6° 14′ 37″. Ans. 14° 32′ 9″.
2. 7 mi. 1 rd. 5 yds. 2 ft. 11 ins., 9 mi. 16 rds. 2 yds. 1 ft .7 ins., and 5 mi. 14 rds. 1 yd. 2 ft. 1 in. Ans. 21 mi. 32 rds. 4 yds. 2 ft. 1 in.
3. What are the contents of three barrels which contain respectively 45 gals. 2 qts., 47 gals. 3 qts., and 44 gals.?

Ans. 2 hhds. 11 gals. 1 qt.

4. How much land is there in three lots which contain respectively 7 A. 89 sq. rds. 7 sq. ft., 11 A. 145 sq. rds. 2 sq. ft., and 9 A. 14 sq. rds.? Ans. 28 A. 88 sq. rds. 1 sq. yd.

EXERCISE VII.

Subtract

1. 18 gals. 3 qts. 1 pt. from 27 gals. 2 qts.

Ans. 8 gals. 2 qts. 1 pt.

2. 3 T. 9 cwt. 81 lbs. 13 oz. from 15 T. 7 cwt. 47 lbs. 1 oz.

Ans. 11 T. 17 cwt. 65 lbs. 4 oz.

Multiply

3. 21 bus. 3 pks. 7 qts. 1 pt. by 41.

Ans. 901 bus. 1 pk. 3 qts. 1 pt.

4. A rapid-fire gun fires 14 shots per minute. Each projectile weighs 3 lbs. 5 oz. What weight of metal is discharged in 1 min. 30 secs.? Ans 69. lbs. 9 oz.

EXERCISE VIII.

Divide

1. 12 gals. 2 qts. 1 pt. 3 gi. by 7. Ans. 1 gal. 3 qts. $2\frac{1}{7}$ gi.
2. 31 lbs. 8 oz. by 17. Ans. 1 lb. $13\frac{11}{17}$ oz.
3. How long will it take to travel 1 mile, at the rate of 74 miles in 9 hrs. 37 mins.? Ans. 7 mins. $47\frac{31}{37}$ secs.

4. How long will it take to pump full a tank which has a capacity of 3700 gals. 3 qts. 1 pt., if the rate of delivery is 19 gals. 1 qt. 1 pt. per second? Ans. 3 mins. $11\frac{2}{155}$ secs.

(Suggestion.—Reduce both divisor and dividend to pints before dividing.)

EXERCISE IX.

1. What is the second power or square of 79? Ans. 6,241
2. What is the third power or cube of 91? Ans. 753,571
3. What is the fifth power of $\frac{3}{7}$? Ans. $\frac{243}{16807}$
4. Find the square of 207. Ans. 42,849
5. Find the cube of .0036. Ans. .000000046656

EXERCISE X.

1. What is the difference between involution and evolution?

Find the square roots of the following:

2. 211.7025 Ans. 14.55
3. $\frac{243}{16807}$ Ans. .3607+

Find the cube root of the following:

4. $\frac{687}{4898}$ Ans. .519+
5. 687543 to two decimal places. Ans. 88.26+

EXERCISE XI.

Solve the following:

1. $\sqrt{737}$ Ans. 27.14+
2. $\sqrt[3]{13498792}$ Ans. 238.10+
3. $\sqrt[4]{37498644}$ Ans. 78.25+
4. $\sqrt{.0034918}$ Ans. .05909+
5. $\sqrt{\frac{823}{41187}}$ Ans. .446+

EXERCISE XII.

What is the ratio of

1. 87 to 261? Ans. 1 : 3
2. $\frac{78}{517}$ to $\frac{61}{78}$? Ans. 1898 : 8479
3. .847 to .0042? Ans. 201.66+

Find the value of x in the following:

4. $2 : 100 :: 17 : x$ Ans. 850
5. $65 : x :: 69 : 850$ Ans. 800.72+
6. $8 \div 3 + 19 : (14 - 7) + 9 :: 16 - (2 \times 7) : x$ Ans. $1.06\frac{3}{8}$

REVIEW QUESTIONS

ON

ELEMENTARY ALGEBRA AND MENSURATION.

EXERCISE I.

1. Find the value of x,

$$6x + 3b - (6 + 4b) - 5 = 4.$$

2. What is the value of x when $b = 4$ and $h = 6$,

$$x = \frac{h\,b^3}{12}$$

3. If $m = 8$, $n = 6$ and $e = 3$, find the value of x from the following equation,

$$-m = n - x - 2e.$$

4. Multiply $3ac - 9bc^2 + 6c$ by $2a - 7c$.

5. Divide $9m^2x^2 + 24mxy + 16y^2$ by $3mx + 4y$.

EXERCISE II.

1. Multiply $3ab + 6c - 1$ by $2c - ab$.

2. Multiply $6mn - 2m + 3n - 1$ by $4n - 3m + 4$.

3. Divide $12a^2x + 8axy + 6ax - 6ay - 4xy^2 - 3y$ by $2ax - y$.

4. Divide $a^3 + 2a^2b + 2ab^2 + b^3$ by $a^2 + ab + b^2$.

5. $\dfrac{(x^3 + y^3)(x - y)}{x^2 - y^2} = ?$

EXERCISE III.

Find the value of x in the following equations.

1. $x = \dfrac{ac + m - n}{b}$, when $a = 3$, $b = 50$, $c = 25$, $m = 26$ and $n = 1$.

2. $x = \frac{(p \times l) - a\,c}{2}$, when $p = 24$, $l = 30$, $a = 6$ and $c = 10$.

3. $a\,x = a\,b - 2\,a\,x + 28$, when $a = 2$, and $b = 16$.

4. If $m = 11$, $a = 900$ and $n = 5$, what is the value of c in the following equation?

$$\sqrt{a} = c\,m - n^2.$$

5. If $a = 3$, $b = 6$ and $c = 5$, find the value of d,

$$\frac{a^2 + 3\,c}{2} = b\,d - (2\,a^2 + a\,b)$$

EXERCISE IV.

1. If I $= 14$ and R $= 5$ what is the value of W from the equation

$$\mathrm{W} = \mathrm{I}^2\mathrm{R}.$$

2. If S $= 9000$, E $= .85$, $p = 115$ and D $= 60$ find t from the formula

$$t = \frac{\mathrm{D}\,p}{2\,\mathrm{S}\,\mathrm{E}}.$$

3. Find the value of E from the formula

$$\mathrm{E} = \frac{\mathrm{W}\,v^2}{2\,g} \text{ if } v = 16,\ \mathrm{W} = 24 \text{ and } g = 32.16.$$

4. If S $= 887.92$ feet and $g = 32.16$ find the value of t.

$$\mathrm{S} = \tfrac{1}{2}\,g\,t^2.$$

5. Find the value of P from the equation

$$\mathrm{P} = \frac{806{,}000\,t^2}{l\,d}.$$

in which $t = .25$, $l = 48$ and $d = 6\frac{1}{2}$.

EXERCISE V.

1. How many square feet in a square 27 rods on each side?

2. The area of a rectangle is 37.8 square feet, and the base is 7 feet. What is the altitude?

3. The diagonal of a trapezium is 21 feet long, the perpendiculars to this diagonal from the vertices on either side are 9 feet and 8 feet. What is the area?

Ans. 178.5 square feet.

4. If the area of a triangle is 500 square yards and the altitude is 180 feet, what is the base?

5. How many square feet are there in a building lot having two parallel sides 140 feet and 116 feet in length and 83 feet apart?

EXERCISE VI.

1. Find the altitude and area of an isosceles triangle whose base is 20 feet long and the equal sides each 26 feet long.

2. The base of a right triangle is 18 feet and the other leg 24 feet. What is the hypothenuse?

3. What is the length of the longest rod that, without bending, can be put into a box 1 yard long, 1 foot wide and 9 inches high, measured on the inside?

Ans. 3 feet 3 inches.

4. The circumference of a circle is 574.9128 feet. What is the area?

5. The area of a circle is 283.5294 square feet. What is the radius?

EXERCISE VII.

1. Find the entire surface of a rectangular solid whose dimensions are 2, 3 and 4 feet.

2. Find the volume of a prism 5 feet high, having for its base a triangle each side of which is 10 feet long.

3. Find the entire surface and the volume of a cylinder 7 inches in diameter and $1\frac{1}{2}$ feet high.

4. The volume of a sphere is 140,501 cubic inches. What is its diameter?

5. The radius of a sphere is 28.5 feet. What is the surface?

EXERCISE VIII.

Miscellaneous Problems.

1. At 3 cents per square foot how much would it cost to paint the outside of a cylindrical tank 14 feet high and 16 feet in diameter? Include the cover.

2. How many cubic feet are there in a piece of timber 15 feet long, 16 inches square at one end and 12 inches square at the other?

Ans. $20\frac{5}{9}$ cu. ft.

3. Find the weight of a hollow steel shaft 12 feet long having an outside diameter of 8 inches and an inside diameter of 4 inches. Assume the steel to weigh 485 pounds per cubic foot.

4. Iron being 7.2 times as heavy as water and water weighing $62\frac{1}{2}$ pounds per cubic foot, what is the weight of an iron ball 6 inches in diameter?

5. Find the weight of water contained in a vertical boiler 4 feet in diameter and 10 feet high, the water level being 8 feet from the bottom. Assume 124 tubes each 2 inches in diameter.

Ans. 4930.5 pounds (approx.)

6. Find the lateral surface of a regular four-sided pyramid 12 feet high, the length of one side of the base being 10 feet.

7. The wheel of a bicycle is 28 inches in diameter. How many times will the wheel turn in going a mile?

8. A house that is 50 feet long and 40 feet wide has a pyramidal roof, whose altitude is 15 feet. Find the length of a rafter reaching from a corner of the building to the vertex of the roof. Ans. 35.35+ ft.

9. The diameter of a boiler is 6 feet, the length 18 feet. If the maximum depth of the water is 4 feet, what is the volume of the steam space?

10. What is the volume of a frustum of a cone, the altitude of which is 9 feet, and the diameters of the bases 11 feet and 8 feet 7 inches? Ans. 681.15+ cu. ft.

11. A gallon of water contains 231 cubic inches. How many gallons are there in a cylindrical cistern 11 feet deep and 7 feet in diameter? Ans. 3,166.7+ gals.

12. What is the amount of surface exposed to the water in the above problem?

13. What is the difference in area between a circle 8 feet in diameter and a regular hexagon inscribed in it?

Ans. 8.69+ sq. ft.

14. Suppose instead of a circle and regular hexagon we had a regular hexagonal prism inside a cylinder, what would be the difference in volume, the altitude of each being 4 feet $7\frac{1}{2}$ inches?

REVIEW QUESTIONS

ON THE SUBJECT OF

ALGEBRA.

EXERCISE I.

Find the value of the following when $a = 3$, $b = 4$, $c = 1$, and $x = 2$.

1. $ab + bx - c$. Ans. 19.
2. $a^3 + b^2 - x$. Ans. 41.
3. $b^a - a^x$. Ans. 55.
4. $\frac{ax}{bc} - \frac{c}{x^2}$. Ans. $\frac{5}{4}$.

EXERCISE II.

1. One number is three times as great as another. The sum of the two is 20. What are the numbers? Ans. 5 and 15.
2. What is the number of degrees difference between 28° and — 5° on a thermometer? Ans. 33°
3. A man has a dollars, and spends b dollars. How may we express his assets?
4. Reduce to simplest form:

$$\frac{\frac{1}{2b} - \frac{1}{3a}}{\frac{3a}{2b} - \frac{2b}{3a}}$$ Ans. $\frac{1}{3a + 2b}$

EXERCISE III.

Remove the parentheses from the following:

1. $a^2 - 14a - (3a + 7)$.
2. $x^2 + 4xy - (7 + 19xy)$.
3. $m^3 - 5m^2n - [17mn^2 - (6n^3 - 8)]$.
4. $a^4 + (a^3 - \overline{4a^2 + a}) - (3a - \overline{9a^3 - 11})$.

EXERCISE IV.

Multiply the following:

1. $m - n$ and $-x$.
2. $x^2 - 4y$ and $x + 3y$. Ans. $x^3 + 3x^2y - 4xy - 12y^2$.
3. $a + 3b$ and $a + b$.
4. $a - 2b - 6c$ and $a + 3c$.

Ans. $a^2 - 2ab - 3ac - 6bc - 18c^2$

EXERCISE V.

Divide the following:

1. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
2. $x^3 + 2x^2y - xy^2 - 2y^3$ by $x + 2y$.
3. $x^4 - 2x^2y^2 + y^4$ by $x - y$.
4. $x^9y^3 - x^8y + x^4y^5 - x^3y^3$ by $x^5 + y^2$.

EXERCISE VI.

Square the following:

1. $a - b$
2. $x - 3y$
3. $3m + 5n$
4. $2a^2 - 13b^2$

EXERCISE VII.

Factor the following:

1. $a^2x - a^2y$
2. $am - an + bm - bn$
3. $x^3 - x^2 + x - 1$
4. $ax + 2bx - ay - 2by$

EXERCISE VIII.

Factor the following:

1. $am^2 - an^2 + bm^2 - bn^2$

Ans. $(a + b)\ (m + n)\ (m - n)$

2. $ax^3 - ay^3$
3. $a^2m - b^2m$
4. $m^2x^4 - m^2y^4 - n^2x^4 + n^2y^4$

Ans. $(m + n)\ (m - n)\ (x^2 + y^2)\ (x + y)\ (x - y)$

EXERCISE IX.

Simplify the following:

1. $\frac{1}{a} + \frac{1}{b}$

REVIEW QUESTIONS

ON THE SUBJECT OF

ALGEBRA.

EXERCISE I.

Find the value of x in the following:

1. $(2x-1)\ (3x+2) = (3x-5)\ (2x+20)$. Ans. 2
2. $\frac{3}{x} - \frac{5}{2x} = 7 - \frac{3}{2x}$ Ans. $\frac{2}{7}$
3. $4x - \frac{2x-3}{3} + \frac{1}{2}(x-9) = 5x$ Ans. -3
4. $\frac{(x+5)^2}{x-3} = \frac{5x+1}{5}$ Ans. -2

EXERCISE II.

Solve:

1. Eleven years ago Mr. A. was 4 times as old as Mr. B., and in 13 years he will be only twice as old. What are their ages at present? Ans. { Mr. A. 59 years. Mr. B. 23 years.

2. The stones which pave a square court would just cover a rectangular area whose length is 6 yards longer, and breadth 4 yards shorter, than the side of the square. Required the area of the court? Ans. 144 square yards.

3. The denominator of a fraction exceeds the numerator by 6; if 8 is added to the denominator, the value of the fraction is $\frac{1}{3}$. Required the fraction? Ans. $\frac{7}{13}$

4. If I were to make my field 5 rods longer and 4 rods wider its area would be increased by 240 square rods; but if I

were to make its length 4 rods less, and its width 5 rods less its area would be diminished by 210 square rods. Required the length, width and area?

Ans. { Length, 30 rods
Width, 20 rods
Area, 600 sq. rods

EXERCISE III.

Solve the following equations:

1. $\begin{cases} 2x - 3y = 4 \\ 6x - y = 28 \end{cases}$ Ans. $\begin{cases} x = 5 \\ y = 2 \end{cases}$

2. $\begin{cases} 2x - 3y = -24 \\ 3x + 2y = 3 \end{cases}$ Ans. $\begin{cases} x = -3 \\ y = 6 \end{cases}$

3. $\begin{cases} 2x - 7y = 8 \\ 4y - 9x = 19 \end{cases}$ Ans. $\begin{cases} x = -3 \\ y = -2 \end{cases}$

4. $\begin{cases} 2x - 5y = -19 \\ 3y + 4z = 13 \\ 2z - 5x = 12 \end{cases}$ Ans. $\begin{cases} x = -2 \\ y = 3 \\ z = 1 \end{cases}$

EXERCISE IV.

1. Find the square root of $169x^4y^8$
2. Find the cube root of $x^6y^9z^3$

Express with radical signs

3. $x^{\frac{1}{2}}$
4. $y^{\frac{3}{5}}$

EXERCISE V.

What is the value of:

1. x^0
2. y^{-3}

Write with positive exponents:

3. $a^3b^{-5}c^{-1}$

What is the value of:

4. $a^5 \times a^{-3}$

EXERCISE VI.

Reduce to simplest form:

1. $\sqrt{\frac{9}{8x^3}}$
2. $\sqrt{\frac{2}{3}}$

3. $\sqrt{4x^2y} + \sqrt{9y^3}$

Reduce to equivalent radicals of same degree:

4. $\sqrt[3]{2}$ and $\sqrt{3}$

EXERCISE VII.

Solve the following:

1. $5x^2 + 5 = 3x^2 + 55$ Ans. $x = \pm 5$
2. $2x^2 + 5x = 3$ Ans. $x = \frac{1}{2}$ or -3
3. The sum of the squares of two numbers is 113. What are the numbers if their sum is 15? Ans. 7 and 8.
4. $\begin{cases} x + y = 1 \\ xy = -6 \end{cases}$ Ans. $\begin{cases} x = 3 \text{ or } -2 \\ y = -2 \text{ or } 3 \end{cases}$

EXERCISE VIII.*

Find the last term and sum of:

1. 1, 6, 11 .... to 15 terms. Ans. $l = 71$ $S = 540$
2. 8, 4, 2 .... to 9 terms. Ans. $l = \frac{1}{32}$ $S = \frac{511}{32}$

Find the value of:

3. .7272. Ans. $\frac{8}{11}$

Find the fifth term of:

4. $(a - \sqrt{a})^{13}$ Ans. $715a^{11}$

*Optional.

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